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OPERATOR SEMIGROUPS ACTING ON A Γ-SEMIGROUPS

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Abstract

The concept of Γ-semigroup is a generalization of semigroups. In this paper, we briefly introduce the action of left (right) operator semigroups on a Γ-semigroup and deduce in particular that there exists an inclusion preserving bijection between the set of all right ideals of S and the set all right ideals of $L \times S$.

Keywords: Γ-semigroup; Operator semigroup.

1 Introduction

The notion of Γ in algebra was first introduced by Nobusawa [8] as a generalization of ring in the form of Γ-ring. Let M and Γ be additive groups such that for all $a, b, c \in M$ and $\gamma, \beta, \alpha \in \Gamma$, we have $a\gamma b \in M$ and $\gamma a \beta \in \Gamma$ for every a, b, γ and β, then M is called a Γ-ring if the following conditions are satisfied:

(i) $(a_1 + a_2)\gamma b = a_1\gamma b + a_2\gamma b$,
$a(\gamma_1 + \gamma_2)b = a\gamma_1 b + a\gamma_2 b$,
$a\gamma(b_1 + b_2) = a\gamma b_1 + a\gamma b_2$,

(ii) $(a\gamma b)\beta c = a\gamma(b\beta c) = a(\gamma b\beta)c$,

(iii) if $a\gamma b = 0$ for any $a, b \in M$, then $\gamma = 0$.

The structure of Γ-rings as initiated by [8] was studied by Barnes [1], Luh [7], Ravishankar and Shukla [9], Buys and Groenewald [3], Booth [2] and Kyuno [6]. Motivated by this generalization of a ring, Sen [13] defined the concept of Γ-semigroup. Later, Sen and Saha [14] redefined the Γ-semigroup by weakening slightly the defining conditions of Γ-semigroup to ensure it preserves semigroup structure. The development of Γ-semigroups hinges on the fact that subsets of a semigroup naturally inherits associativity but not necessarily closed. As a result of this, various generalizations and analogues of corresponding results in semigroup theory have been obtained based on the modified definition (see [10, 15, 16, 17]).

In an attempt to broaden the theoretical aspect of Γ-semigroup theory, Dutta and Adhikari [4] slightly changed the defining conditions of Γ-semigroup by Sen and Saha [14] and then introduced the notion of left operator semigroup and right operator semigroup of a Γ-semigroup.

In [4], the authors described the relationship between the set of Γ-ideals and operator semigroups. In relation to the concept, Dutta and Chattopadhyay [5] initiated the notions of uniformly strongly prime semigroup, uniformly strongly prime ideal, Rees congruence on Γ-semigroup and uniformly strongly prime Γ-semigroup and studied these via operator semigroups of Γ-semigroup. Sardar *et al.* [12] showed that the left operator and right

operator semigroups of a Γ-semigroup with unities are Morita equivalence monoid and further established that there is a close connection between the Morita equivalence of monoids and Γ-semigroups. Although there is significantly number of published results in literature on operator semigroups of a Γ-semigroup, however the aspect of operator semigroups acting on a Γ-semigroup observed by [11] has not been given much attention. This serves as a motivation to write this paper and we deduce some results.

2 Preliminaries

We recall some definitions and results related to this paper.

Definition 2.1 *Let S and Γ be two non-empty sets. Then S is called a Γ-semigroup if there exist mappings $S \times \Gamma \times S \longrightarrow S \mid (a, \alpha, b) \longrightarrow a\alpha b \in S$ and $\Gamma \times S \times \Gamma \longrightarrow \Gamma \mid (\alpha, a, \beta) \longrightarrow \alpha a \beta \in \Gamma$ satisfying the identities $a\alpha(b\beta c) = a(\alpha b\beta)c = (a\alpha b)\beta c$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.*

The modified definition of Γ-semigroup by Sen and Saha [14] may be regarded as one-sided Γ-semigroup.

Definition 2.2 *Let $S = \{a, b, c...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then S is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \longrightarrow S \mid (a, \alpha, b) \longrightarrow a\alpha b \in S$ satisfying the property $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.*

A Γ-semigroup S is called commutative if $x\alpha y = y\alpha x$ for every $x, y \in S$ and $\alpha \in \Gamma$.

Let A and B be two subsets of a Γ-semigroup S. We define the set

$$A\Gamma B = \{a\gamma b \mid a \in A,\ b \in B \text{ and } \gamma \in \Gamma\}.$$

For simplicity we write $a\Gamma B$, $A\Gamma b$ and $A\gamma B$ instead of $\{a\}\Gamma B$, $A\Gamma\{b\}$ and $A\{\gamma\}B$ respectively.

Let S be an arbitrary semigroup and Γ be a non-empty set. Define a mapping $S \times \Gamma \times S \longrightarrow S$ by $a\alpha b = ab$ for all $a, b \in S$ and $\alpha \in \Gamma$. It is easy to see that S is a Γ-semigroup. Thus, a semigroup can be considered as a Γ-semigroup.

In the following, some examples of Γ-semigroups are presented.

Example 2.1 *Let $S = Z$ be the set of all integers and $\Gamma = \{n \mid n \in N\}$. Then S is a Γ-semigroup with the operation defined by $a\alpha b = a + \alpha + b$ for all $a, b \in S$ and $\alpha \in \Gamma$.*

Example 2.2 *Let S be a set of all negative rational numbers. Obviously S is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} \mid p$ is prime$\}$. Let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b; then $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence S is a Γ-semigroup.*

Example 2.3 *Let $S = \{-i, i, 0\}$ and $\Gamma = S$. Then S is a Γ-semigroup with respect to multiplication of complex numbers whereas S does not reduce to semigroup with respect to multiplication of complex numbers.*

The following definitions and theorems can be found in [5] except Definitions 2.3.

Definition 2.3 *Let S be a Γ-semigroup. A nonempty subset A of S is called left (right) Γ-ideal of S if $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$). Further, a non-empty A of a Γ-semigroup S is called Γ-ideal if A is both a left and a right Γ-ideal of S.*

Definition 2.4 *Let S be a Γ-semigroup. Let L and R be the left and right operator semigroups of the Γ-semigroup S. If there exist an element $[e, \delta] \in L$ ($[\delta, e] \in R$) such that $e\delta x = x$ ($x\delta e = x$) for all $x \in S$, then S is said to have the left unity $[e, \delta]$ (right unity $[\delta, e]$).*

Definition 2.5 *Let S be a Γ-semigroup. We define a relation ρ on $S \times \Gamma$ as follows:*

$$(x, \alpha)\rho(y, \beta) \Longleftrightarrow x\alpha s = y\beta s, \ \forall \ s \in S.$$

Obviously ρ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α). Let $L = \{[x, \alpha] : x \in S, \ \alpha \in \Gamma\}$. Then L is a semigroup with respect to multiplication defined by $[x, \alpha][y, \beta] = [x\alpha y, \beta]$. The semigroup L is called the left operator semigroup of S. Similarly, the right operator semigroup R of a Γ-semigroup S is defined as $R = \{[\alpha, x] : \alpha \in \Gamma, \ x \in S\}$, where $[\alpha, x][\beta, y] = [\alpha, x\beta y]$, for all $x, y \in S$ and $\alpha, \beta \in \Gamma$.

Example 2.4 *Using Example 2.3,*

$S \times \Gamma = \{(-i,-i),(0,0),(i,i),(-i,0),(-i,i),(0,-i),(0,i),(i,-i),(i,0)\}$
and let $\rho = \{(-i,-i),(0,0),(i,i),(-i,i),(i,-i)\}$ *be a relation on* $S \times \Gamma$.

By routine calculation, it is obvious that ρ *is an equivalence relation. The equivalence class* $[x,\alpha] = \{(y,\beta) \in \rho \mid (y,\beta)\rho(x,\alpha)\}$. *Therefore,*

$$\begin{aligned}
[-i,-i] &= \{(-i,-i),(i,i)\} \\
[0,0] &= \{(0,0)\} \\
[i,i] &= \{(i,i),(-i,-i)\} \\
[-i,i] &= \{(-i,i),(i,-i)\} \\
[i,-i] &= \{(i,-i),(i,-i)\}
\end{aligned}$$

$\Longrightarrow [-i,-i] = [i,i]$ *and* $[-i,i] = [i,-i]$ *and the set form* $L = \{[-i,-i],[i,i],[-i,i],[i,-i]\}$ *is a left operator semigroup of* S. *Similarly, the right operator semigroup* R *of* S *can be obtained.*

Theorem 2.1 *Let* S *be a* Γ*-semigroup. If* $[e,\delta]$ *is left unity of* S, *then* $[e,\delta]$ *is the identity element of* L.

Theorem 2.2 *Let* S *be a* Γ*-semigroup. If* $[\mu,f]$ *is right unity of* S, *then* $[\mu,f]$ *is the identity element of* R.

Theorem 2.3 *Let* S *be a* Γ*-semigroup with left and right unities and* L *be its left operator semigroup.*

(i) If Q *is a* Γ*-ideal of* S, *then* $Q^{+'}$ *is a* Γ*-ideal of* L.

(ii) If P *is a* Γ*-ideal of* L, *then* P^{+} *is a* Γ*-ideal of* S.

Theorem 2.4 *Let* S *be a* Γ*-semigroup with left and right unities and* R *be its right operator semigroup.*

(i) If Q *is a* Γ*-ideal of* S, *then* $Q^{+'}$ *is a* Γ*-ideal of* R.

(ii) If P *is a* Γ*-ideal of* R, *then* P^{*} *is a* Γ*-ideal of* S.

Theorem 2.5 *Let* S *be a* Γ*-semigroup with left and right unities and let* L *and* R *be its left operator semigroup and right operator semigroup respectively. Then there is an inclusion preserving bijection between the set of all* Γ*-ideals of* S *and the set of all* Γ*-ideals of* $L(R)$ *via the mapping* $Q \longrightarrow Q^{+'}$ $(Q \longrightarrow Q^{*'})$ *where* Q *is a* Γ*-ideal of* S.

3 Main Results

Throughout S stands for one-sided Γ-semigroup unless otherwise mentioned.

The following proposition and theorem show the commutativity and isomorphism of operator semigroups.

Proposition 3.1 *Let S be a commutative Γ-semigroup and $\alpha \in \Gamma$. Then the left operator semigroup L and the right operator semigroup R of S are commutative.*

Proof
The proof is straightforward.

Theorem 3.1 *If S is a commutative Γ-semigroup and $\alpha \in \Gamma$, then the operator semigroup L and the right operator semigroup R of S are isomorphic.*

Proof
Define $f : L \longrightarrow R$ by $f([a,\alpha]) = [\alpha,a]$. Let $[a,\alpha] = [b,\alpha]$. Then $a\alpha s = b\alpha s$ for all $s \in S$. Since S is commutative, $s\alpha a = s\alpha b$ for all $s \in S$. So, $[\alpha, a] = [\alpha, b]$. Thus, f is well-defined. The mapping is injective, since

$$\begin{aligned} [\alpha, a] = [\alpha, b] &\Longrightarrow s\alpha a = s\alpha b \ \ \forall\ s \in S \\ &\Longrightarrow a\alpha s = b\alpha s \\ &\Longrightarrow [a,\alpha] = [b, \alpha] \end{aligned}$$

Again, f is surjective, since for any $[\alpha, a] \in R$, $a \in S$ and $\alpha \in \Gamma$, we have $[\alpha, a] = f[a,\alpha]$.

Also, for all $a, b \in S$ and $\alpha \in \Gamma$,

$$\begin{aligned} f([a, \alpha][b, \alpha]) = f([a\alpha b,\alpha]) = [\alpha, a\alpha b] &= [\alpha, b\alpha a] \\ &= [\alpha,b][\alpha,a] \\ &= [\alpha, a][\alpha, b] \\ &= f([a,\alpha])\ f([b\alpha]) \end{aligned}$$

So, f is a homomorphism. Therefore, L and R are isomorphic.

In the following we consider operator semigroups acting on a Γ-semigroup.

Definition 3.1 *Let S be a Γ-semigroup and its left and right operator semigroup are respectively L and R. Then for every $a \in S$ and $\alpha \in \Gamma$, we define $L \times S \longrightarrow S$ and $S \times R \longrightarrow S$ respectively as follow:*

$$[a, \alpha]s := a\alpha s \text{ and } s[\alpha, a] := s\alpha a.$$

The following remark follows from Definition 3.1.

Remark 3.1 *If S is a commutative Γ-semigroup, then $L \times S = S \times R$.*

Example 3.1 *Clearly, Example 2.3 shows that S is commutative and from Example 2.4 it is easy to verify that $L \times S = S \times R$ for every $s \in S$ and $\alpha \in \Gamma$.*

Proposition 3.2 *If A is an ideal of S, then $L \times A$ is an ideal of $L \times S$.*

Proof
Suppose that A is an ideal of S and $[a, \alpha]t \in L \times A$. Then for every $s, t \in S$ and $\beta \in \Gamma$, we have $a\alpha(s\beta t) \in A$. Thus, $[a, \alpha][s, \beta]t = [a\alpha s, \beta]t \in L \times A$. Similarly, we can prove that $[s, \beta][a, \alpha]t = [s\beta a, \alpha]t \in L \times A$. Hence, $L \times A$ is an ideal of $L \times S$.

Theorem 3.2 *There exists an inclusion preserving bijection between the set of all right ideals of S and the set all right ideals of $L \times S$.*

Proof
Suppose that $\mathcal{I}(S)$ and $\mathcal{I}(L \times S)$ sre the sets of all right ideals of S and $L \times S$ respectively. Clearly, the mapping $f : \mathcal{I}(S) \longrightarrow \mathcal{I}(L \times S)$ by $f(I) = L \times I$ is well-defined. Since I is a right ideal of S, $I\Gamma S \subseteq I$. Thus, $I \subseteq L \times I$. On the other hand, since S has a left unity, $L \times I \subseteq (L \times I)\Gamma S \subseteq I$. Thus, $L \times I = I$. Hence, f is bijective. It remains to show that f is an inclusion preserving mapping. Let I and J be two right ideals of S such that $I \subseteq J$. We have to show that $L \times I \subseteq L \times J$. Let $[a, \alpha] \in L$. Then for every $s \in S$ we have $s \in I$ and so $[a, \alpha]s \in L \times I$. Since $I \subseteq J$, we have $s \in J$. Hence, $[a, \alpha]s \in L \times J$. Therefore, $L \times I \subseteq L \times J$. Hence the proof.

Remark 3.2 *Similar characterisations can be proved for right operator semigroup acting on a Γ-semigroup and right Γ-ideal.*

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AN APPLICATION OF SMITH NORMAL FORM

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Abstract

Modules are a generalization of the vector spaces of linear algebra over a ring instead of over a field of scalars. Some of the results in linear algebra are extremely useful in studying modules. In this paper, we highlight the use of Smith normal form in studying finitely generated modules over a univariate polynomial ring. As an application, we take advantage of Smith normal form to understand the structure of certain syzygy modules.

Keywords: Module; Basis; Smith Normal Form; PID.

1 Introduction

Linear algebra deals with vector spaces and linear transformations over a field such as real or complex numbers. Linear algebra is very well understood, serves as a fundamental tool for mathematical economics, data science, machine learning, and has many applications, from mathematical physics to modern algebra and coding theory.

Modules are a generalization of the vector spaces of linear algebra over a ring instead of over a field of scalars. A fundamental fact of linear algebra over a field is a finitely generated vector space has a basis – the minimal generating (spanning) set of a vector space are linearly independent and therefore form a basis. However, modules are more complicated than vector spaces; for instance, not all modules have a basis. Only a special family of modules called *free modules* have a basis. It remains an interesting and an active research area to find the minimal generating set, or an upper bound for the minimal generating set for different kinds of modules under various of conditions [2], [3], [4], and [6].

In linear algebra, matrix factorization is a mathematical technique used to decompose a matrix into the product of multiple matrices. There are various ways to decompose a matrix depending on the context in which that matrix is used. For instance, in computer science and machine learning, singular value decomposition is one of the most important computational method. Whereas, representing a matrix in the Smith normal form has many computational applications in number theory, group theory, and homological algebra.

In this short paper, we focus on the Smith normal form, and its applications in modules over univariate polynomial rings $\mathbb{K}[x]$ with $\mathbb{K}$ an algebraically closed field of characteristic zero. Our goal is to find a minimal set of generators for certain syzygy modules using Smith normal form.

Recall that given a generating set $\{f_1, \ldots, f_m\}$ of an ideal (or a module) over a ring R, a relation or the first *syzygy* between the generators is an m-tuple $(a_1, \ldots, a_m) \in R^m$ such that

$$a_1 f_1 + a_2 f_2 + \cdots a_m f_m \equiv 0.$$

The set of syzygies form a *syzygy module*, denoted by $S = \mathrm{Syz}(f_1, \dots, f_m)$. A *Koszul syzygy* is one of the form

$$(f_j)f_i + (-f_i)f_j = 0, \text{ for } i \neq j.$$

The Koszul syzygies generate a submodule of the syzygy module S.

Our attention is centered on the submodule K that are generated by the "Koszul-like" forms over a univariate polynomial ring $\mathbb{K}[x]$:

$$\left[0, \cdots, 0, -\frac{f_j}{\gcd(f_i, f_j)}, 0, \cdots, 0, \frac{f_i}{\gcd(f_i, f_j)}, 0, \cdots, 0\right]^T, \ 1 \leq i < j \leq m.$$

We seek the answers to the question that how the minimal sets of generators for K and S are related.

This paper is structured as the following. We begin in Section 2 with a brief review of results concerning the Smith normal form over Principal Ideal Domains (PIDs), and the Hilbert–Burch Theorem. We then study the minimal set of generators for the submodule K of a syzygy module S in Section 3. The main results of this paper are Theorems 3.1 and 3.2, where we provide an explicit equation to describe a relationship between a basis for K and a basis for S. We flush out our theorems by a simple and illustrative example.

2 A Brief Review

The Smith Normal Form (SNF) is a canonical form to which a matrix can be transformed using elementary row and column operations. It is particularly useful when dealing with matrices whose entries come from a ring with special properties, such as PIDs. Theorem 2.1 is a well-known results concerning SNF over PID.

Theorem 2.1. *(Smith Normal Form over PID [1, Theorem 3.1]) Let R be a PID and let $A \in M_{m,n}(R)$. Then there is a $U \in GL(m, R)$ and a $V \in GL(n, R)$ such that*

$$UAV = \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix},$$

where $r = \operatorname{rank}(A)$, and D is a diagonal matrix with diagonal entries $s_1, s_2, \ldots, s_r$ with $s_i \neq 0$ for $i = 1, \ldots, r$, and $s_i \mid s_{i+1}$ for $i = 1, \ldots, r-1$. Additionally, s_i's are unique up to multiplication by a unit and are called the elementary divisors, invariants, or invariant factors, which can be computed (up to multiplication by a unit) as $s_i = \frac{d_i(A)}{d_{i-1}(A)}$, where $d_i(A)$ is called i-th determinant divisor that equals the greatest common divisor of the determinants of all $i \times i$ minors of the matrix A and $d_0(A) := 1$. Furthermore, if R is a Euclidean domain, the matrices U, V can be taken to be a product of elementary matrices.

In the context of modules (generalizations of vector spaces) over a PID, the SNF reveals important information about the structure of the module and its relationship to other modules. Throughout this paper, we should use Theorem 2.1 to investigate the minimal set of generators for modules. To our advantage, we refer SNF as the matrix factorization that decomposes A into the product of invertible matrices with a diagonal matrix, and write $A = U \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix} V$ for some invertible matrices U, V.

For the convenience of our readers, we also cite a part of the Hilbert–Burch Theorem below. The complete statement of the Hilbert–Burch Theorem is beyond the scope of this paper, we only cite the portion of the theorem that concerns the first syzygy module $\operatorname{Syz}(f_1, \ldots, f_m)$ over $\mathbb{K}[x]$.

Theorem 2.2. *(Hilbert–Burch Theorem [5]) If M is a module minimal generated by m elements over a polynomial ring $\mathbb{K}[x]$ over a field $\mathbb{K}$, then the first syzygy module of M is always a free module of rank $m-1$.*

We want to emphasize that this result states that the first syzygy module $\operatorname{Syz}(f_1, \ldots, f_m)$ has a basis consisting of $m-1$ linearly independent generators over univariate polynomial rings; however, it is not true in general for multivariate polynomial rings.

3 Main Results

Theorem 3.1. *Let $\{p_1, \ldots, p_{m-1}\}$ be a basis for the syzygy module $S = \operatorname{Syz}(f_1, \ldots, f_m)$ where $f_1, \ldots, f_m \in R = \mathbb{K}[x]$. Let K be the submodule*

generated by the syzygies of the form

$$\left[0,\cdots,0,-\frac{f_j}{\gcd(f_i,f_j)},0,\cdots,0,\frac{f_i}{\gcd(f_i,f_j)},0,\cdots,0\right]^T,\ 1\le i<j\le m.$$

Then

1. *K is minimally generated by $m-1$ syzygies of such form, that is,* $\mathrm{rank}(K)=m-1$.
2. *There exist at most $m-1$ distinct $C_1,\ldots,C_{m-1}\in R$ where C_i is the smallest degree element in R such that $C_ip_i\in K$.*
3. *The colon ideal $\langle K:S\rangle=\{r\in R\mid rS\subseteq K\}$ is the principle ideal generated by $C=\mathrm{LCM}(C_1,\ldots,C_{m-1})$, the least common multiple of $C_1,\ldots,C_{m-1}$, that is, $\langle K:S\rangle=\langle\mathrm{LCM}(C_1,\ldots,C_{m-1})\rangle=\langle C\rangle$,*
4. *$S/K\cong R/\langle t_1\rangle\times R/\langle t_2\rangle\times\cdots\times R/\langle t_{m-1}\rangle$, where $t_1,\ldots,t_{m-1}$ are the elementary divisors of T for some T such that $K=ST$.*

Proof. Let K be a syzygy submodule generated by the columns of the matrix

$$\begin{bmatrix}
\frac{-f_m}{\gcd(f_1,f_m)} & \frac{-f_{m-1}}{\gcd(f_1,f_{m-1})} & \cdots & 0 & 0 & \cdots & 0\\
0 & 0 & \cdots & \frac{-f_m}{\gcd(f_2,f_m)} & \frac{-f_{m-1}}{\gcd(f_2,f_{m-1})} & \cdots & 0\\
0 & 0 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & \frac{f_1}{\gcd(f_1,f_{m-1})} & \cdots & 0 & \frac{f_2}{\gcd(f_2,f_{m-1})} & \cdots & \frac{-f_m}{\gcd(f_m,f_{m-1})}\\
\frac{f_1}{\gcd(f_1,f_m)} & 0 & \cdots & \frac{f_2}{\gcd(f_2,f_m)} & 0 & \cdots & \frac{f_{m-1}}{\gcd(f_m,f_{m-1})}
\end{bmatrix}$$

This matrix is of size $m\times l$ where $l=\binom{m}{2}$. To simplify the the matrix expression, let K_i for $i=1,\ldots,l$ be the i-th column of the above matrix.

Proof of item 1. It is easy to observe that the $(m-1)\times(m-1)$ submatrix formed by the first $m-1$ columns and the last $m-1$ rows has a non-zero determinant. Hence the rank of this matrix is at least $m-1$, i.e., $\mathrm{rank}(K)\ge m-1$. On the other hand, since K is a submodule of S, $\mathrm{rank}(K)\le\mathrm{rank}(S)$, and $\mathrm{rank}(S)=m-1$ by Theorem 2.2. Thus, $\mathrm{rank}(K)=m-1$.

Proof of item 2. Without loss of generality, we may select $m-1$ columns of the matrix that generate the submodule K, and name these generators $K_1, K_2, \ldots, K_{m-1}$. Let $a = [a_1, \ldots, a_m]^T \in S$. Since the matrix $[K_1, K_2, \ldots, K_{m-1}] \in M_{m,m-1}(R)$ is of rank $m-1$, the matrix equation $[K_1, \ldots, K_{m-1}]X = a$ has a unique solution over the function field of R, i.e.

$$X = \left[b_1/c_1, b_2/c_2, \cdots, b_{m-1}/c_{m-1}\right]^T \in (\mathbb{K}(x))^{m-1}, \text{ with } \gcd(b_i, c_i) = 1.$$

Let $c = \mathrm{LCM}(c_1, c_2, \ldots, c_{m-1}) \in \mathbb{K}[x]$, set $c_i' = \frac{c}{c_i}$ for $i = 1, \ldots, m-1$. Then $c_i' \in \mathbb{K}[x]$, $cX = [c_1'b_1, c_2'b_2, \ldots, c_{m-1}'b_{m-1}]^T \in (\mathbb{K}[x])^{m-1}$, and

$$\begin{aligned} c\,a &= c([K_1, \ldots, K_{m-1}]X) = [K_1, \ldots, K_{m-1}](cX) \\ &= \left[K_1, K_2, \ldots, K_{m-1}\right] \begin{bmatrix} c_1'b_1 \\ c_2'b_2 \\ \vdots \\ c_{m-1}'b_{m-1} \end{bmatrix} = \sum_{i=1}^{m-1} (c_i'b_i)K_i \in K. \end{aligned}$$

Note since $c = \mathrm{LCM}(c_1, c_2, \ldots, c_{m-1})$, up to a constant multiple, c is the smallest degree element in R such that $ca \in K$. That is, for any factor β of c that is not a constant multiple of c, $\beta a \notin K$.

Since $\{p_1, \ldots, p_{m-1}\}$ is a basis for S, by the above argument, there exists at most $m-1$ distinct $C_1, C_2, \ldots, C_{m-1} \in R$ where C_i is the smallest degree element in R such that $C_i p_i \in K$ for each i.

Proof of item 3. Let $C = \mathrm{LCM}\,(C_1, \ldots, C_{m-1})$, then $C_i p_i \in K$ for each i yields that $Cp_i \in K$. Thus, for any $a \in S$, $a = \beta_1 p_1 + \cdots \beta_{m-1} p_{m-1}$ for some $\beta_1, \ldots, \beta_{m-1} \in R$, and

$$Ca = C(\beta_1 p_1 + \cdots + \beta_{m-1} p_{m-1}) = \beta_1(Cp_1) + \cdots + \beta_{m-1}(Cp_{m-1}) \in K.$$

Therefore, we have $\langle K : S \rangle = \langle C \rangle = \langle \mathrm{LCM}\,(C_1, \ldots, C_{m-1}) \rangle$.

Proof of item 4. Since K is a submodule of S, so the generators $K_1, \ldots, K_{m-1}$ of K can be expressed as $K_i = \sum_{j=1}^{m-1} \beta_{ji} p_j$ for some $\beta_{ji} \in R$. Thus, $K = ST$ where $T = [\beta_{ji}]_{i,j=1,\ldots,m-1} \in M_{m-1,m-1}$.

By Theorem 2.1, $T = UDV$ for some invertible $U, V \in M_{m-1,m-1}$ and

$$D = \begin{bmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & t_{m-1} \end{bmatrix} \text{ where } t_i \text{ are the elementary divisors of } T.$$

$$K = ST = SUDV \quad \Longleftrightarrow \quad KV^{-1} = (SU)D.$$

Since the columns of $KV^{-1} = [K'_1, \ldots, K'_{m-1}]$ is another basis for K; and similarly, columns of $SU = [p'_1, \ldots, p'_{m-1}]$ is another basis for S. Thus, we have the equality of the matrices

$$[K'_1, \ldots, K'_{m-1}] = KV^{-1} = (SU)D = [t_1 p'_1, \ldots, t_{m-1} p'_{m-1}].$$

Thus consider the following map

$$\phi : \begin{array}{ccc} R \times R \times \cdots \times R & \to & (SU)/(KV^{-1}) \cong S/K \\ (r_1, r_2, \ldots, r_{m-1}) & \to & r_1 p'_1 + \cdots + r_{m-1} p'_{m-1}. \end{array}$$

We see that

$$\begin{aligned} \ker(\phi) &= \{(r_1, \ldots, r_{m-1}) \in R^{m-1} \mid \sum_{i=1}^{m-1} r_i p'_i \in KV^{-1}\} \\ &= \langle t_1 \rangle \times \langle t_2 \rangle \times \cdots \langle t_{m-1} \rangle. \end{aligned}$$

Therefore,

$$S/K \cong (SU)/(KV^{-1}) \cong R/\langle t_1 \rangle \times R/\langle t_2 \rangle \times \cdots \times R/\langle t_{m-1} \rangle.$$

□

We want to emphasize that Theorem 3.1 is true under the condition that the polynomial ring is univariate, and not true in general for multivariate polynomial rings.

Next, we will continue with the notations used in Theorem 3.1, and show that SNF stated in Theorem 2.1 can be used to identify a some special properties of syzygy modules.

Theorem 3.2. *Let $S = [p_1, \ldots, p_{m-1}] \in M_{m,m-1}$ whose columns are the minimal set of generators of the syzygy module S of $f_1, \ldots, f_m$; and $K = [K_1, \ldots, K_{m-1}]$ whose columns are the minimal set of generators for the "Koszul-like" syzygy submodule K over the univariable polynomial ring $R = \mathbb{K}[x]$ obtained in Theorem 3.1. Let $K = UDV$ be the SNF for K. Then the last row U_m of the matrix U^{-1} is a constant multiples of $f_1, \ldots, f_m$.*

Proof. First, we express part 2 of Theorem 3.1 in term the following matrix equation:

$$
\begin{aligned}
[C_1p_1, \ldots, C_{m-1}p_{m-1}] &= [p_1, \ldots, p_{m-1}] \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & C_{m-1} \end{bmatrix} \\
&= KQ \text{ for some } Q \in M_{m-1,m-1} \\
&= UDVQ = U \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{m-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix} VQ
\end{aligned}
$$

where U, V are invertible $U \in M_{m,m}$, $V \in M_{m-1,m-1}$, $D \in M_{m,m-1}$.

Multiplying both sides by U^{-1} yields

$$
U^{-1}[C_1p_1, C_2p_2, \ldots, C_{m-1}p_{m-1}] = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{m-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix} VQ,
$$

and the last row of this matrix equation gives

$$
\begin{aligned}
& U_m[C_1p_1, C_2p_2, \ldots, C_{m-1}p_{m-1}] = [0, 0, \ldots, 0] \\
\Longrightarrow\ & U_m(C_ip_i) = C_i(U_m p_i) = 0 \quad \Longrightarrow \quad U_mp_i = 0, \ \forall i = 1, \ldots, m-1 \\
\Longrightarrow\ & U_m \text{ is orthogonal to } p_i, \ \forall i = 1, \ldots, m-1.
\end{aligned}
$$

Since $\{p_1, p_2, \ldots, p_{m-1}\}$ is a minimal set of generators for the syzygy module of $f_1, \ldots, f_m$, we must have $[f_1, \ldots, f_m]$ is orthogonal to each p_i for $i = 1, \ldots, m-1$. Since $S = [p_1, \ldots, p_{m-1}]$ and $\operatorname{rank}(S) = m-1$, the rank of null space of S is one. Hence $U_m = \beta[f_1, \ldots, f_m]$ for some $\beta \in R = \mathbb{K}[x]$. Now, the fact that U is invertible over R implies that $\det(U) \in \mathbb{K}$. Therefore, $\beta \in \mathbb{K}$, otherwise $\det(U)$ would have a non-constant polynomial as a factor, contradicting to the fact that U is invertible.

Thus, we conclude that the last row U_m of the matrix U^{-1} is a constant multiples of $f_1, \ldots, f_m$.

□

We shall use the following example to illustrate the the results in Theorem 3.1 and Theorem 3.2.

Example 3.3. *Consider* $[f_1, f_2, f_3] = [1, x^2-1, x^3+1]$. *One can compute a minimal set of generators for the syzygy module* $\operatorname{Syz}(f_1, f_2, f_3)$ *is formed by the columns of the matrix* $S = [p_1, p_2] = \begin{bmatrix} x^2-1 & x^3+1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$.

The submodule K *is generated by the syzygies in the columns of the matrix*

$$[K_1, K_2, K_3] = \begin{bmatrix} x^2-1 & x^3+1 & 0 \\ -1 & 0 & x^2-x+1 \\ 0 & -1 & -(x-1) \end{bmatrix}.$$

Since $K_3 = -(x^2-x+1)K_1 + (x-1)K_2$, *the submodule* K *is minimally generated by* K_1, K_2. *We will write* $K = [K_1, K_2]$. *We note that* $K_1 = p_1$, *and* $-xK_1 + K_2 = p_2$, *that is*

$$\begin{aligned} [K_1, K_2] \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} x^2-1 & x^3+1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x^2-1 & x+1 \\ -1 & x \\ 0 & -1 \end{bmatrix} \\ &= [p_1, p_2]. \end{aligned}$$

Following the notation of Theorem 3.1, $C_1 = C_2 = 1$, *and therefore,*

$$\langle K : S \rangle = \langle \mathrm{LCM}(C_1, C_2) \rangle = \langle 1 \rangle. \quad \Longrightarrow \quad S \cong K.$$

This can also be verified as the following. Since $\begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$,

$$\begin{aligned} [K_1, K_2] &= \begin{bmatrix} x^2-1 & x^3+1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} x^2-1 & x+1 \\ -1 & x \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}^{-1} \\ &= [p_1, p_2]T = ST \ \text{ where } T = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

In terms of Theorem 3.1 (4.), the elementary divisors of T *are:* $t_1 = t_2 = 1$. *Hence* $S/K \cong R/\langle 1\rangle \times R/\langle 1\rangle \cong 0$, *that is,* $S \cong K$.

Now, compute the SNF of K

$$\begin{aligned} K &= UDV = \begin{bmatrix} x^2-1 & x^3+1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{where} \\ U &= \begin{bmatrix} x^2-1 & x^3+1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \\ U^{-1} &= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & x^2-1 & x^3+1 \end{bmatrix}. \end{aligned}$$

The last row of the matrix U^{-1} *is* $U_3 = [1, x^2-1, x^3+1] = [f_1, f_2, f_3]$, *which verifies the result of Theorem 3.2.*

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ABSTRACT PETER-WEYL THEORY FOR SEMICOMPLETE ORTHONORMAL SETS

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Abstract

The central concept in the harmonic analysis of a compact group is the completeness of Peter-Weyl orthonormal basis as constructed from the matrix coefficients of a maximal set of irreducible unitary representations of the group, leading ultimately to the direct sum decomposition of its L^2- space. A Peter-Weyl theory for a semicomplete orthonormal set is also pos- sible and is here developed in this paper for compact groups. Existence of semicomplete orthonormal sets on a compact group is proved by an explicit construction of the standard Riemann- Lebesgue semicomplete orthonormal set on the Torus, T. This approach gives an insight into the role played by the L^2- space of a compact group, which is discovered to be just an example (indeed the largest example for every semicomplete orthonormal set) of what is called a prime-Parseval subspace, which we proved to be dense in the usual L^2- space, serves as the natural domain of the Fourier transform and breaks up into a direct-sum decomposition. This paper essentially gives the harmonic analysis of the prime-Parseval subspace of a compact group corresponding to any semicomplete orthonormal set, with an introduction to what is expected for all connected semisimple Lie groups through the notion of a $K-$semicomplete orthonormal set.

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§1. Introduction.

Harmonic analysis on a compact group is mainly a direct consequence of the famous *Peter-Weyl theory* which gives a consistent method, via the computation of the matrix coefficients of its *irreducible unitary representations,* of deriving a *complete orthonormal set* which is immediately responsible for the direct-sum decomposition of its L^2- space and *regular representation.* Even though such a complete orthonormal set is non-existence for non-compact topological groups and hence the harmonic analysis on *non-compact topological groups,* as we know for *connected nilpotent* and *semisimple Lie groups,* has had to be developed through other means notably via the differential equations satisfied by the (*spherical*) functions derived as matrix coefficients of irreducible unitary representations constructed from *parabolic* and *cohomological inductions* and the completeness afforded by the *Plancherel theorem* (which in the final analysis still depends on the availability and properties of the *discrete series* (known to be the irreducible unitary representations corresponding to some complete orthonormal set) of some distinguished compact subgroups), it still found to be appropriate (and to have a sense of finality) to have some forms of *Peter-Weyl* results on such *non-compact topological groups.*

It is however possible to get at the decomposition of the regular representation of a compact group G (for a start) via the indirect use of the notion of a *semicomplete orthonormal set* on such a group, leading to the consideration of a distinguished subspace of $L^2(G)$ which is established to be *topologically dense.* The study in this paper opens up this field of research by a detailed look at the *compact case.* The paper is arranged as follows.

§2. contains a quick review of the well-known notion of a complete orthonormal set on a compact group, giving the detailed of the aforementioned consistent way of constructing such a set through *Peter-Weyl theorem* which then leads to the direct-sum decomposition of its L^2-space. The concept of a *semicomplete orthonormal set* on a compact group G is introduced in §3. with constructible examples (prominent among which is the *Riemann-Lebesgue* orthonormal set)on the Torus, $\mathbb{T}$, where we derived and used the properties of the *Fourier* and *prime-Parseval subspaces* of $L^2(G)$. Chief among these properties is the topological denseness of every *prime-Parseval subspace* in $L^2(G)$. This takes us to the *Fourier transform* of the *prime-Parseval subspace* and its

direct-sum decomposition into *invariant subspaces.* The last section gives an introductory extension of the results of §3. on compact groups to connected semisimple Lie groups with finite center.

§2. Fourier and Parseval subspaces for complete orthonormal set.

A mutually orthonormal family $\{\chi_\alpha\}_{\alpha\in A}$ in a Hilbert space, $(H, \langle\cdot,\cdot\rangle)$ is said to be *complete* (in H) if $x \in H$ is such that $\langle x, \chi_\alpha\rangle = 0$ (for every $\alpha \in A$) implies $x = 0$. This means that a family $\{\chi_\alpha\}_{\alpha\in A}$ of mutually orthonormal members of H is complete whenever it can be shown that the zero element of H is the *only* non-member of the family that is mutually orthonormal to all members of the said family. Two other equivalent methods of confirming the completeness of the family $\{\chi_\alpha\}_{\alpha\in A}$ are as follows.

2.1 Lemma. ([5.], p. 3) *Let $\{\chi_\alpha\}_{\alpha\in A}$ denote a mutually orthonormal family in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. The following are equivalent:*
(a) Every $x \in H$ can be expressed as $x = \sum_{\alpha\in A}\langle x, \chi_\alpha\rangle\chi_\alpha$.
(b) Every $x \in H$ satisfies $\| x \|^2 = \sum_{\alpha\in A} | \langle x, \chi_\alpha\rangle |^2$.
(c) $\{\chi_\alpha\}_{\alpha\in A}$ is complete in H. □

The informed reader would observe that (a) of (2.1) is a *Fourier series* expansion of x while (b) of (2.1) is its *Parseval equality,* both with respect to $\{\chi_\alpha\}_{\alpha\in A}$. The import of this equivalence (in the light of (a) of (2.1) (respectively, (b) of (2.1))) is that every $x \in H$ has a Fourier series expansion in terms of any known complete orthonormal family in H. We could then say that the subset $H(\chi_\alpha)$ of H given as

$$\{x \in H : x = \sum_{\alpha\in A}\langle x, \chi_\alpha\rangle\chi_\alpha, \text{ for some orthonormal family } \{\chi_\alpha\}_{\alpha\in A} \text{ in } H\}$$

(equivalently, the subset $H_{\mathfrak{P}}(\chi_\alpha)$ of H given also as

$$\{x \in H : \| x \|^2 = \sum_{\alpha\in A} | \langle x, \chi_\alpha\rangle |^2, \text{ for some orthonormal family } \{\chi_\alpha\}_{\alpha\in A} \text{ in } H\})$$

is exactly H if, and only if, $\{\chi_\alpha\}_{\alpha\in A}$ is complete. Indeed another version of the equivalence of Lemma 2.1, whose formulation serves as our point of departure, is given as follows.

2.2 Lemma. *Let $\{\chi_\alpha\}_{\alpha\in A}$ denote a mutually orthonormal family in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. The following are equivalent:*
(a) $H(\chi_\alpha) = H$
(b) $H_{\mathfrak{P}}(\chi_\alpha) = H$
(c) $\{\chi_\alpha\}_{\alpha\in A}$ is complete in H. □

2.3 Remarks. It may be safely conjectured that the *Fourier subspace* $H(\chi_\alpha)$ as well as the *Parseval subspace* $H_{\mathfrak{P}}(\chi_\alpha)$ (of a Hilbert space H) with respect to a complete mutually orthonormal family will always be equal to H. It will be a delight to study the disparity between the *Fourier subspace* $H(\chi_\alpha)$ as well as the *Parseval subspace* $H_{\mathfrak{P}}(\chi_\alpha)$ (of H with respect to the mutually orthonormal family $\{\chi_\alpha\}_{\alpha\in A}$) and their inclusions in H, when the family $\{\chi_\alpha\}_{\alpha\in A}$ is not complete.

For example, if the family $\{\chi_\alpha\}_{\alpha\in A}$ of mutually orthonormal members in H is such that $\langle x,\chi_\alpha\rangle = 0$ *(for every $\alpha \in A$) does not necessarily imply whether $x = 0$ or $x \neq 0$*, it possible to then have that

$$0 \leq \| x \|^2 = \sum_{\alpha\in A} | \langle x,\chi_\alpha\rangle |^2 = 0,$$

showing in this case (for the family $\{\chi_\alpha\}_{\alpha\in A}$ in which $\langle x,\chi_\alpha\rangle = 0$ (for every $\alpha \in A$) does not necessarily imply whether $x = 0$ or $x \neq 0$) that we now have $H_{\mathfrak{P}}(\chi_\alpha) = \{0\}$ $(= H(\chi_\alpha) \neq H$, showing that both subspaces are too small and far from being equal to H). This shows at a glance the importance of completeness of the family $\{\chi_\alpha\}_{\alpha\in A}$ in the consideration of the *Parseval equality*, for the non-triviality of these two subspaces $H(\chi_\alpha)$ and $H_{\mathfrak{P}}(\chi_\alpha)$ and for the sustenance of the relationship of equality (of Lemma 2.2) between $H(\chi_\alpha)$ and $H_{\mathfrak{P}}(\chi_\alpha)$. □

However, and as it shall be shown in the next section, these two subspaces, $H(\chi_\alpha)$ and $H_{\mathfrak{P}}(\chi_\alpha)$ may be considered for an *appropriately chosen* not-necessarily complete orthonormal family $\{\chi_\alpha\}_{\alpha\in A}$ and with which they would still be found not to be too small in sizes (in comparison with H). This choice of a not-necessarily complete orthonormal family $\{\chi_\alpha\}_{\alpha\in A}$ would equally help and be appropriate in order that both $H(\chi_\alpha)$ and $H_{\mathfrak{P}}(\chi_\alpha)$ be *lifted* to all of H. All this in a moment.

A well-known method of computing complete orthonormal family of functions is via the matrix coefficients of irreducible unitary representations of a compact groups G which is then used to decompose $L^2(G)$ into invariant subspaces, leading to the decomposition of the right regular representation on G (which sadly, does not generalize to *non-compact topological groups*). Here is the technique.

Denote the *dual* of a compact group G by $\widehat{G}$, consisting of all its equivalence classes of irreducible unitary representations. For $\lambda \in \widehat{G}$ denote by u^λ_{ij} the corresponding matrix coefficient representative of the class λ whose

degree is also denoted by $d(\lambda)$. Then the set

$$\{\sqrt{d(\lambda)}u_{ij}^{\lambda} : \lambda \in \widehat{G},\ 1 \leq i,j \leq d(\lambda)\}$$

consists of a maximal set of complete orthonormal family of functions in $L^2(G)$ and (hence) every $f \in L^2(G)$ can be expanded as

$$f = \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j}^{d(\lambda)} \langle f, u_{ij}^{\lambda} \rangle u_{ij}^{\lambda}$$

(with convergence in the norm of $L^2(G)$) whose *Fourier transform*

$$\widehat{f} : \widehat{G} \to M_{d(\lambda)}(\mathbb{C}) : \lambda \mapsto \widehat{f}(\lambda) = (\widehat{f}(\lambda)_{ij})_{i,j=1}^{d(\lambda)}$$

is given as $\widehat{f}(\lambda)_{ij} := \langle f, u_{ij}^{\lambda} \rangle$ (where $M_{d(\lambda)}(\mathbb{C})$ denotes the algebra of matrices with entries in $\mathbb{C}$ and degree $d(\lambda)$). It then follows that for any compact group G, the *Fourier subspace* $L^2(G)(\sqrt{d(\lambda)}u_{ij}^{\lambda})$ of $L^2(G)$ is given as

$$L^2(G)(\sqrt{d(\lambda)}u_{ij}^{\lambda}) := \{f \in L^2(G) : f = \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j}^{d(\lambda)} \langle f, u_{ij}^{\lambda} \rangle u_{ij}^{\lambda}\} = L^2(G)$$

$(=L^2(G)_{\mathfrak{P}}(\sqrt{d(\lambda)}u_{ij}^{\lambda})$, the *Parseval subspace* of $L^2(G)$), with respect to the family $\{\sqrt{d(\lambda)}u_{ij}^{\lambda} : \lambda \in \widehat{G},\ 1 \leq i,j \leq d(\lambda)\}$. We then have the abstract direct-sum decomposition of $L^2(G)$ given as

$$L^2(G) = \bigoplus_{\lambda \in \widehat{G}} \bigoplus_{i=1}^{d(\lambda)} H_i^{\lambda},$$

where $H_i^{\lambda} := \sum_{j=1}^{d(\lambda)} \mathbb{C}u_{ij}^{\lambda}$. This is the content of *Peter-Weyl Theorem,* [5.], and we shall refer to the set

$$\{\sqrt{d(\lambda)}u_{ij}^{\lambda} : \lambda \in \widehat{G},\ 1 \leq i,j \leq d(\lambda)\}$$

as the *standard Peter-Weyl orthonormal set* on G.

The inability of being able to get an orthonormal family in $L^2(G)$ for a non-compact topological group G in the above tradition of Peter-Weyl is the first stumbling block to harmonic analysis on such groups, which has been

considerably understood and completely developed via a rigorous treatment of the rich structure of differential equations satisfied by matrix-coefficients of members of each of the classes in $\widehat{G}$, [2]. This paper presents a constructive method of getting a not-necessarily complete orthonormal set which is *close enough* to being a complete orthonormal family in an arbitrary Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and/or in $L^2(G)$, for a compact group (and introduced the same technique for a semisimple Lie group) offering a more general Fourier series expansion of each member of an appropriate subspace of H and/or $L^2(G)$.

Starting with a compact group (before extending the notion to all connected semisimple Lie groups, with finite center, via its *Iwasawa decomposition*) we would however not approach harmonic analysis on the groups via the completeness (and consequent denseness) of the *standard Peter-Weyl orthonormal set,* but via a denseness in the L^2–space which would be found to be possible from an *almost complete* orthonormal set.

§3. Semicomplete orthonormal set in a compact group.

The existence of different special functions and polynomials of mathematical physics, which have been established to be orthonormal in various semisimple Lie groups (compact and non-compact types), is well-known. However the absence of completeness of these orthornormal families (under the structure of their individual corresponding groups) is the first stumbling block to a direct *Peter-Weyl harmonic analysis* of them. In this section we shall define and study the concept of a *semicomplete* orthonormal family in a compact group in order to extend this concept to the harmonic analysis of *all* semisimple Lie groups in the next section.

3.1 Definition. (*Semicomplete orthonormal family*) *Let G denote a compact group and let the members of the non-empty set A be ordered such that $A = \{\alpha_i^j\}_{i,j}$. An orthonormal family $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ in $L^2(G)$ is said to be semicomplete if given $\epsilon > 0$ there exist some non-zero scalars*

$$\gamma_1, \cdots, \gamma_k, \cdots, \beta_{11}, \cdots, \beta_{ij}, \cdots \in \mathbb{C}$$

and $n \in \mathbb{N}$ such that

$$\| \sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^\lambda\rangle u_{ij}^\lambda - \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j\in A} \langle f, \chi_{\alpha_i^j}\rangle \chi_{\alpha_i^j} \|_2 < \epsilon$$

for every $f \in L^2(G)$. □

The quantity

$$\sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^{\lambda}\rangle u_{ij}^{\lambda}$$

in Definition 3.1 above may be replaced with f (due to the *Peter-Weyl Theorem*), so that the other quantity

$$\sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j\in A} \langle f, \chi_{\alpha_i^j}\rangle \chi_{\alpha_i^j}$$

(in the same Definition above) should be seen as the *total contribution of* $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ *in* $L^2(G)$ *in its bid to attain* f. Thus the informed reader would see that the inequality in Definition 3.1 above simply gives a measure of how close to the completeness (of $\{\sqrt{d(\lambda)}u_{ij}^{\lambda}\}$) is the orthonormal set $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$.

The *standard Peter-Weyl orthonormal basis* $\{\sqrt{d(\lambda)}u_{ij}^{\lambda}\}$ used in the above Definition 3.1 may be replaced by any other known complete orthonormal set $\{v_\mu\}_{\mu\in B}$ in $L^2(G)$ while the concept of a semicomplete orthonormal set (for $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$) could also be defined for an arbitrary Hilbert space, H, so as to have what may be generally called *a semicomplete orthonormal set in* H *with respect to (the complete orthonormal set)* $\{v_\mu\}_{\mu\in B}$ *in* H. If in this general case the set $\{v_\mu\}_{\mu\in B}$ in H is also not necessarily complete, we may arrive at the notion of *a relative semicomplete orthonormal set* for $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ in H with respect to $\{v_\mu\}_{\mu\in B}$ in H. Thus Definition 3.1 may therefore be seen as giving *semicompleteness of* $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ *in* $L^2(G)$ *with respect to the standard Peter-Weyl orthonormal basis* $\{\sqrt{d(\lambda)}u_{ij}^{\lambda}\}$.

It is clear that every complete orthonormal set in $L^2(G)$ (or in any Hilbert space, H) is automatically semicomplete; simply choose $A=\widehat{G}$, $\gamma_j=\beta_{ij}=1$, but not conversely. An inductive method of immediately constructing a semicomplete orthonormal set in a compact group is by a method of *selective omission* of some number of members in any known complete (or of the *standard Peter-Weyl*) orthonormal set with a *controlled bound*. The control of the bound in the method of *selective omission* would be achieved using the *Riemann-Lebesgue Lemma.*

This method, as contained in the following, equally gives an *existence* argument for the concept of a semicomplete orthonormal set in a compact group.

3.2 Lemma. (*Existence of a semicomplete orthonormal set: the standard Riemann-Lebesgue orthonormal set on the Torus,* $\mathbb{T}$) *There exist* $\lambda_0 \in \widehat{\mathbb{T}}$ *for which*

$$| \langle f, u^{\lambda}_{km} \rangle |< \frac{\epsilon}{d(\lambda_0)^2},$$

for every $f \in L^2(\mathbb{T})$, $| \lambda | \geq | \lambda_0 |$ *and* $1 \leq k, m \leq d(\lambda_0)$. *Moreover,*

$$\{\sqrt{d(\lambda)}u^{\lambda}_{ij} :\ \lambda \in \widehat{G} \setminus \{\lambda_0\},\ 1 \leq i,j \leq d(\lambda)\}$$

is a semicomplete orthonormal set on $\mathbb{T}$.

Proof. Since the dual group $\widehat{\mathbb{T}}$ is discrete, so that

$$\lim_{|\lambda| \to \infty} \langle f, u^{\lambda}_{ij} \rangle = \lim_{|\lambda| \to \infty} \widehat{f(\lambda)}_{ij} = 0 \text{ (by the \textit{Riemann-Lebesgue Lemma})},$$

it follows that there are (infinitely) many possible $\lambda \in \widehat{G}$ (choose such one λ_0) with $| \lambda | \geq | \lambda_0 |$ for which $| \langle f, u^{\lambda}_{km} \rangle | = | \langle f, u^{\lambda}_{km} \rangle - 0 | < \frac{\epsilon}{d(\lambda_0)^2}$, for every $f \in L^2(\mathbb{T})$ and $1 \leq k, m \leq d(\lambda_0)$, as required. Hence,

$$\| \sum_{\lambda \in \widehat{\mathbb{T}}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u^{\lambda}_{ij} \rangle u^{\lambda}_{ij} - \sum_{\lambda \in \widehat{\mathbb{T}} \setminus \{\lambda_0\}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u^{\lambda}_{ij} \rangle u^{\lambda}_{ij} \|_2 = \| d(\lambda_0) \sum_{i,j=1}^{d(\lambda_o)} \langle f, u^{\lambda_0}_{ij} \rangle u^{\lambda_0}_{ij} \|_2$$

$\leq d(\lambda_0) \sum_{i,j=1}^{d(\lambda_0)} | \langle f, u^{\lambda_0}_{ij} \rangle | < \epsilon$, for every $f \in L^2(\mathbb{T})$. □

The technique of Lemma 3.2 may be extended as follows. Generally, choose (as assured by the *Riemann-Lebesgue Lemma*) $\lambda_0^{(1)}, \lambda_0^{(2)}, \cdots \in \widehat{\mathbb{T}}$ for which

$$\sum_{k=1}^{\infty} | \langle f, u^{\lambda}_{ij} \rangle | < \frac{\epsilon}{(\sum_{k=1}^{\infty} d(\lambda_0^{(k)}))^2}$$

where $| \lambda | \geq \max\{| \lambda_0^{(1)} |, | \lambda_0^{(2)} |, \cdots\}$ and $f \in L^2(\mathbb{T})$. Then, with proof essentially the same as in Lemma 3.2, the set

$$\{\sqrt{d(\lambda)}u^{\lambda}_{ij} :\ \lambda \in \widehat{G} \setminus \{\lambda_0^{(1)}, \lambda_0^{(2)}, \cdots\},\ 1 \leq i,j \leq d(\lambda)\}$$

is a semicomplete orthonormal set on $\mathbb{T}$. We shall henceforth refer to the semicomplete orthonormal set

$$\{\sqrt{d(\lambda)}u^{\lambda}_{ij} :\ \lambda \in \widehat{\mathbb{T}} \setminus \{\lambda_0^{(1)}, \lambda_0^{(2)}, \cdots\},\ 1 \leq i,j \leq d(\lambda)\}$$

as the *standard Riemann-Lebesgue (semicomplete) orthonormal set* on $\mathbb{T}$ (being in correspondence with the *standard Peter-Weyl (complete) orthonormal set,* $\{\sqrt{d(\lambda)}u_{ij}^{\lambda}\}$.)

Other *non-standard* examples of Definition 3.1 may be deduced from the numerous *special functions* of mathematical physics where their corresponding non-zero scalars γ_j and β_{ij} in Definition 3.1 could be calculated from.

3.3 Remarks. In contrast to the zero-subspace $H_{\mathfrak{P}}(\chi_\alpha)$ of Remarks 2.3 we may, in the context of a semicomplete orthonormal set $\{\chi_\alpha\}_{\alpha\in A}$ in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, consider the subspace

$$H'_{\mathfrak{P}}(\chi_\alpha) := \{x \in H : \ \langle x, \chi_\alpha\rangle = 0, \ (\text{for every } \alpha \in A) \text{ implies } x = 0\},$$

for some orthonormal set $\{\chi_\alpha\}_{\alpha\in A}$ in H. It is clear (from Lemma 2.2) that $H'_{\mathfrak{P}}(\chi_\alpha) = H$ (hence equal to $H(\chi_\alpha)$ and $H_{\mathfrak{P}}(\chi_\alpha)$) if, and only if, $\{\chi_\alpha\}_{\alpha\in A}$ is complete in H and that, when $\{\chi_\alpha\}_{\alpha\in A}$ is semicomplete in H or in $L^2(G)$, both $H_{\mathfrak{P}}(\chi_\alpha)$ and $H'_{\mathfrak{P}}(\chi_\alpha)$ are non-zero: an example may be seen from using the *standard Riemann-Lebesgue orthonormal set* on $\mathbb{T}$. In general, we have the following.

3.4 Lemma. *Let* $(H, \langle\cdot,\cdot\rangle)$ *denote any Hilbert space. Then*

$$H(\chi_\alpha) \subseteq H'_{\mathfrak{P}}(\chi_\alpha)$$

for any semicomplete orthonormal set $\{\chi_\alpha\}_{\alpha\in A}$ *in* H.

Proof. Choose any $x \in H(\chi_\alpha)$, then $x = \sum_{\alpha\in A}\langle x, \chi_\alpha\rangle\chi_\alpha$. Now if $\langle x, \chi_\alpha\rangle = 0$, for every $\alpha \in A$, then

$$x = \sum_{\alpha\in A}\langle x, \chi_\alpha\rangle\chi_\alpha = \sum_{\alpha\in A}(0)\chi_\alpha = 0;$$

showing that $x = 0$ as required. □

We shall refer to $H'_{\mathfrak{P}}(\chi_\alpha)$ as the *prime-Parseval subspace* of H and the choice of this term is further reinforced by the following facts.

3.5 Lemma. (*cf.* Lemma 2.2) *Let* $\{\chi_\alpha\}_{\alpha\in A}$ *denote a semicomplete orthonormal set in a Hilbert space* $(H, \langle\cdot,\cdot\rangle)$ *and let* $x \in H$. *Then* $x \in H'_{\mathfrak{P}}(\chi_\alpha)$ *whenever* $\| x \|^2 = \sum_{\alpha\in A} | \langle x, \chi_\alpha\rangle |^2$.

Proof. If $\| x \|^2 = \sum_{\alpha\in A} | \langle x, \chi_\alpha\rangle |^2$ and $| \langle x, \chi_\alpha\rangle | = 0$ (for every $\alpha \in A$), then $\| x \|^2 = \sum_{\alpha\in A} | \langle x, \chi_\alpha\rangle |^2 = \sum_{\alpha\in A}(0) = 0$; showing that $x = 0$. Hence $x \in H'_{\mathfrak{P}}(\chi_\alpha)$. □

Lemma 3.5 shows the first partial connection between the satisfaction of *Parseval equality,* on one hand, and membership in the *prime-Parseval*

subspace, on the other. The last Lemma may also be seen as saying that the subset of H given as

$$\{x \in H : \| x \|^2 = \sum_{\alpha \in A} | \langle x, \chi_\alpha \rangle |^2, \text{ for any orthonormal set } \{\chi_\alpha\}_{\alpha \in A}\}$$

is also a subset of $H'_{\mathfrak{P}}(\chi_\alpha)$, with clear equality when $\{\chi_\alpha\}_{\alpha \in A}$ is complete. It will be satisfying to also have the reverse inclusion,

$$H'_{\mathfrak{P}}(\chi_\alpha) \subseteq \{x \in H : \| x \|^2 = \sum_{\alpha \in A} | \langle x, \chi_\alpha \rangle |^2, \text{ for any orthonormal set } \{\chi_\alpha\}_{\alpha \in A}\}$$

due to the importance of the Parseval equality in the fine properties of Fourier transform. We shall deal with this concern in Lemma 3.12.

Even though a semicomplete orthonormal set $\{\chi_\alpha\}_{\alpha \in A}$ in $L^2(G)$ (or in a Hilbert space $(H, \langle, \cdot, \rangle)$) may not be dense, as it is generally expected of a complete orthonormal set, we may still however employ this orthonormal set to construct some dense subspaces of $L^2(G)$ (or of a Hilbert space $(H, \langle, \cdot, \rangle)$) as follows. Indeed, the following results on the *Fourier subspace* for $L^2(G)$ are also valid for an arbitrary Hilbert space, $(H, \langle, \cdot, \rangle)$ and for a *relative semicomplete orthonormal set* in H.

3.6 Theorem. *Let G denote a compact and let $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ denote a semicomplete orthonormal set on G. Then $L^2(G)(\chi_{\alpha_i^j})$ is topologically dense in $L^2(G)$.*

Proof. Since every $f \in L^2(G)$ may be expanded as

$$\sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^\lambda \rangle u_{ij}^\lambda$$

(with convergence in the norm of $L^2(G)$) it follows that for $\epsilon > 0$ we have

$$\| f - \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^\lambda \rangle u_{ij}^\lambda \|_2 < \frac{\epsilon}{2}.$$

Now

$$\| f - \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} \langle f, \chi_{\alpha_i^j} \rangle \chi_{\alpha_i^j} \|_2 \leq \| f - \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^\lambda \rangle u_{ij}^\lambda \|_2$$

$$+ \| \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^{\lambda} \rangle u_{ij}^{\lambda} - \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} \langle f, \chi_{\alpha_i^j} \rangle \chi_{\alpha_i^j} \|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \square$$

In more specific terms we have the following.

3.7 Corollary. *Let G denote a compact group and let $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ denote a semicomplete orthonormal set on G. Then every $f \in L^2(G)$ can be expanded as*

$$f = \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} \langle f, \chi_{\alpha_i^j} \rangle \chi_{\alpha_i^j}$$

for some $\gamma_j, \beta_{ij} \in \mathbb{C}$ with convergence in the norm on $L^2(G)$. $\square$

We may refer to the expansion of f in Corollary 3.7 as a *semi-Fourier series expansion* for $f \in L^2(G)$ or H with respect to $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$. A stronger form of Theorem 3.6 carved in the form of the equivalence of Lemma 2.2 and which generalizes the fact that a mutually orthonormal family $\{\chi_\alpha\}_{\alpha \in A}$ is complete (in a Hilbert space $(H, \langle \cdot, \cdot \rangle)$) if, and only if, $H(\chi_\alpha) = H$ (*cf.* Lemma 2.2) is also possible when the mutually orthonormal family $\{\chi_\alpha\}_{\alpha \in A}$ is semicomplete in H. We prove this below in the special case of $H = L^2(G)$.

3.8 Theorem. *Let G denote a compact group and let $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ denote a mutually orthonormal set on G whose Fourier subspace is denoted as $L^2(G)(\chi_{\alpha_i^j})$. Then $L^2(G)(\chi_{\alpha_i^j})$ is topologically dense in $L^2(G)$ if, and only if, $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ is semicomplete.*

Proof. That $L^2(G)(\chi_{\alpha_i^j})$ is topologically dense in $L^2(G)$ if $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ is semicomplete is the content of Theorem 3.6. Now choose $f \in L^2(G)$, then

$$\| \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^{\lambda} \rangle u_{ij}^{\lambda} - \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} \langle f, \chi_{\alpha_i^j} \rangle \chi_{\alpha_i^j} \|_2$$

$$\leq \| \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^{\lambda} \rangle u_{ij}^{\lambda} - f \|_2 + \| f - \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} \langle f, \chi_{\alpha_i^j} \rangle \chi_{\alpha_i^j} \|_2$$

$\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$(using the *Peter-Weyl theorem* and Corollary 3.7, respectively). $\square$

This Theorem would enable us to see the *Peter-Weyl series expansion* of every $f \in L^2(G)$, given as

$$f = \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^{\lambda} \rangle u_{ij}^{\lambda}$$

(with convergence in the L^2–norm), as the restriction of the *semi-Fourier series expansion*

$$f = \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} \langle f, \chi_{\alpha_i^j} \rangle \chi_{\alpha_i^j}$$

to the *standard Peter-Weyl (complete) mutually orthonormal set* $\{\sqrt{d(\lambda)}u_{ij}^{\lambda}\}$. Indeed Theorem 3.8 leads to the same conclusion for the *prime-Parseval subspace* $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$.

3.9 Corollary. *Let G denote a compact group and let $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ denote a mutually orthonormal set on G. Then $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ is topologically dense in $L^2(G)$ if, and only if, $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ is semicomplete.*

Proof. Consider Lemma 3.4 in the light of Theorem 3.8. □

The inclusion $L^2(G)(\chi_{\alpha_i^j}) \subseteq L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ of Lemma 3.4, when combined with both Theorem 3.7 and Corollary 3.9, implies the following.

3.10 Corollary. *$L^2(G)(\chi_{\alpha_i^j})$ is topologically dense in $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$.* □

The converse of Lemma 3.5 is now immediate for both $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ and (even) $H'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ in any arbitrary Hilbert space, $(H, \langle \cdot, \cdot \rangle)$.

3.11 Lemma.(*cf.* Lemma 2.2) *Let G denote a compact group and let $\{\chi_\alpha\}_{\alpha \in A}$ denote a mutually orthonormal set on G. Then $f \in L^2(G)'_{\mathfrak{P}}(\chi_\alpha)$ if, and only if, $\| f \|_2^2 = \sum_{\alpha \in A} | \langle f, \chi_\alpha \rangle |^2$.*

Proof. Let $f \in L^2(G)'_{\mathfrak{P}}(\chi_\alpha)$. We may take $f \in L^2(G)(\chi_\alpha)$ due to Corollary 3.10; so that $f = \sum_{\alpha \in A} \langle f, \chi_\alpha \rangle \chi_\alpha$. Hence

$$0 = \| f - \sum_{\alpha \in A} \langle f, \chi_\alpha \rangle \chi_\alpha \|_2^2 = \| f \|_2^2 - \sum_{\alpha \in A} | \langle f, \chi_\alpha \rangle |^2,$$

as required. □

Hence, the *prime-Parseval subspace* $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ may finally be seen (for some orthonormal set $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$) as

$$L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j}) = \{f \in L^2(G) : \| f \|_2^2 = \sum_{\alpha_i^j \in A} | \langle f, \chi_{\alpha_i^j} \rangle |^2\}$$

We now have enough preparation to introduce a Fourier transform $f \mapsto \widehat{f}$ on the *prime-Parseval subspace,* $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$.

Consider $f \in L^2(G)$ and for every $\alpha \in A$ define the matrix $\widehat{f}(\alpha)$ whose entries are given as

$$\widehat{f}(\alpha)_{ij} := \widehat{f}(\alpha_i^j).$$

That is, $\widehat{f}(\alpha)_{ij} := \langle f, \chi_{\alpha_i^j} \rangle$, for $1 \leq i,j \leq n$. The Parseval inequality of $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ (in Lemma 3.11) therefore becomes $\| f \|_2^2 = \sum_{\alpha \in A} \| \widehat{f}(\alpha) \|^2$, for every $f \in L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$, where $\| \widehat{f}(\alpha) \|^2$ is the Hilbert-Schimdt norm of the matrix

$$\widehat{f}(\alpha) = (\widehat{f}(\alpha)_{ij})_{i,j=1}^n = (\widehat{f}(\alpha_i^j))_{i,j=1}^n.$$

In other words, and in terms of our choice of indexing A, we have

$$\| f \|_2^2 = \sum_{i,j=1}^n \sum_{\alpha_i^j \in A} \| \widehat{f}(\alpha_i^j) \|^2,$$

for $f \in L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$.

3.12 Definition. Set $L^2(A)$ as the space of matrix-valued functions φ on A with values in $\bigcup_{n=1}^\infty M_n(\mathbb{C})$ satisfying
(*i*) $\varphi(\alpha_i^j) \in M_n(\mathbb{C})$ for all $\alpha_i^j \in A$ and
(*ii*) $\sum_{i,j=1}^n \sum_{\alpha_i^j \in A} \| \varphi(\alpha_i^j) \|^2 < \infty$. □

The inner product $(\cdot,\cdot)$ on $L^2(A)$ given as

$$(\varphi, \psi) := \sum_{i,j=1}^n \sum_{\alpha_i^j \in A} tr(\varphi(\alpha_i^j)\psi(\alpha_i^j)^*),$$

$\varphi, \psi \in L^2(A)$ converts $(L^2(A), (\cdot,\cdot))$ into a Hilbert space. We can then establish a connection between the *prime-Parseval subspace* $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ (which is a Hilbert subspace of $L^2(G)$) and $L^2(A)$.

3.13 Theorem. (Fourier image of the *prime-Parseval subspace*) *Let G denote a compact group and let $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ denote a semicomplete mutually orthonormal set on G. Then the map*

$$\mathcal{H} : L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j}) \to L^2(A) : f \mapsto \mathcal{H}(f) := \widehat{f}$$

is an isometry of $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ onto $L^2(A)$. □

Theorem 3.13 is very familiar when the semicomplete mutually orthonormal set $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ is the complete mutually orthonormal set $\{\sqrt{d(\lambda)} u_{ij}^\lambda\}$. We do not yet know the general connection between the set A and the dual group $\widehat{G}$, except in the special cases of the *standard Riemann-Lebesgue (semicomplete) orthonormal sets* on $\mathbb{T}$. We however see A as a general form of $\widehat{G}$ which

may take the usual form of $\widehat{G}$ in specific cases. If we set

$$H_i^\alpha := \sum_{j=1}^{n} \mathbb{C}\chi_{\alpha_i^j},$$

for $\alpha = \alpha_i^j \in A$ and $i \in \{1, \cdots, n\}$, then the Hilbert subspace $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ of $L^2(G)$ has the direct-sum decomposition

$$L^2(G)'_{\mathfrak{P}}(\chi_\alpha) = \bigoplus_{\alpha \in A} \bigoplus_{i=1}^{n} H_i^\alpha.$$

The results of this section laid a foundation for harmonic analysis of the *prime-Parseval subspace* $H'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ with respect to a semicomplete orthonormal set $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ in a Hilbert space, H. Having considered the case of the Hilbert space $L^2(G)$, for a compact group G, in this section it will a delight to use these foundational results (on both $H'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ and $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$) in the understanding of further properties of $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ in the full sight of the semicompleteness of $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$. We shall give a very short introduction to this type of study for a connected semisimple Lie group in the next section.

It is clear from Lemma 3.2, for *standard (Riemann-Lebesgue)* examples of a semicomplete orthonormal set in an arbitrary Hilbert space $(H, \langle\cdot,\cdot\rangle)$ or in $L^2(G)$, that the non-zero constants γ_j and β_{ij} would always be $\gamma_j = \beta_{ij} = 1$ for $1 \leq i, j \leq \mid \widehat{G} \setminus \{\lambda_0^{(1)}, \lambda_0^{(2)}, \cdots\} \mid$. However, for *non-standard* examples of a semicomplete orthonormal set in an arbitrary Hilbert space $(H, \langle\cdot,\cdot\rangle)$ or in $L^2(G)$, the *semi-Fourier series expansion* of Corollary 3, 7 may have to be broken down in order for general expressions for γ_j and β_{ij} to be known. A first result along this line is the following.

3.14 Lemma. *Let $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ denote a semicomplete orthonormal set in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and let $x \in H$. Then*

$$\langle x, \chi_{\alpha_i^i}\rangle = \gamma_i \beta_{ii} \langle x, \chi_{\alpha_i^i}\rangle,$$

for $1 \leq i \leq n$. In particular, $\gamma_i\beta_{ii} = 1$.

Proof. We have that $\langle x, \chi_{\alpha_k^l}\rangle = \sum_{j=1}^n \gamma_j \sum_{i=1}^n \beta_{ij} \sum_{\alpha_i^j \in A} \langle x, \chi_{\alpha_i^j}\rangle \langle \chi_{\alpha_i^j}, \chi_{\alpha_k^l}\rangle$. Due to the orthogonality of the set $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ the above equality reduces to $\langle x, \chi_{\alpha_i^i}\rangle = \gamma_i\beta_{ii}\langle x, \chi_{\alpha_i^i}\rangle$, for $1 \leq i \leq n$ as required.

Now $(1 - \gamma_i\beta_{ii})\langle x, \chi_{\alpha_i^i}\rangle = 0$ from where we have $\gamma_i\beta_{ii} = 1$. $\square$

§4. K-semicomplete orthonormal set in a semisimple Lie group.

The success in §3. of the use of the notion of a *semicomplete orthonormal* set in the harmonic analysis of a compact group, culminating in the extraction and elucidation of the *prime-Parseval subspace* as well as its Fourier image, shows the central importance and the correct us of *Parseval equality* and the concept of *completeness* (of an orthonormal set) in the abstract Peter-Weyl theory of a compact group and in the understanding of the hitherto unknown subspaces of $L^2(G)$ under the influence of the Fourier transform. This study (which led us to the consideration of the *prime-Parseval subspace* $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ corresponding to a semicomplete orthonormal set $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ on G) is reminiscence of and may be compared with the extraction and harmonic analysis of the *Schwartz algebra* in the L^2-theory of semisimple Lie groups which was started in the Yale thesis [1(*a.*)] of James Arthur (continued and completed in two later manuscripts, [1(*b.*)] and [1(*c.*)]). In a more recent publication, harmonic analysis of other spaces of functions on semisimple Lie groups, namely of the space of *spherical convolutions,* has been introduced in [3.] leading to the explicit construction of the corresponding *Plancherel formula* for such functions. The present paper has also introduced the *Fourier* and *prime-Parseval subspaces* of $L^2(G)$ (or of any arbitrary Hilbert space, $(H, \langle\cdot,\cdot\rangle)$).

Having shown in §3. the essential importance of the Parseval equality (which is the precursor of the Plancherel formula) in the consideration of the actual subspace of $L^2(G)$ under the natural action of the Fourier transform, we shall here consider studying the same theory (of a semicomplete orthonormal set) but for all semisimple Lie groups, having removed the impediments posed by the *completeness* for orthonormal sets on such Lie groups.

It is well-known that orthonormal sets (of functions and polynomials) are numerous and readily available in the L^2-space (and more recently in some distinguished subspaces of the $L^{2n}-$spaces [4.]) of semisimple Lie groups. Indeed every semisimple Lie group has its corresponding orthonormal set, an example is $G = SL(2,\mathbb{R})$ and its *Legendre functions.*

Even though these sets of orthonormal functions and polynomials are central to harmonic analysis on these groups, their direct importance in or contribution to the decomposition of (sub-)spaces of $L^2(G)$ or expansion of their members is not yet known. In the outlook of the present section (and

of the entire paper) any orthonormal set on a semisimple Lie group known to have been $K-$*semicomplete* (in the sense to be soon made precise) could be a basis of some subspaces of $L^2(G)$.

4.1 Definition. ($K-$semicomplete orthonormal set) *Let $G = KAN$ denote the Iwasawa decomposition of a connected semisimple Lie group G with finite center. An orthonormal set $\{\chi_\alpha\}_{\alpha\in A}$ on G is said to be $K-$semicomplete whenever its restriction to K, written as $\{(\chi_\alpha)_{|K}\}_{\alpha\in A}$, is a semicomplete orthonormal set in $L^2(K)$.* $\Box$

It is relatively easy to construct a $K-$*semicomplete orthonormal set* on any connected semisimple Lie group G, from any given semicomplete orthonormal set on K as follows.

4.2 An example. Choose any of the numerous orthonormal sets $\{\xi_\alpha\}_{\alpha\in A}$ in $L^2(K)$ as constructed in §3. and, for every $x = kan \in G$, define the map $\chi_\alpha : G \to \mathbb{C}$ as

$$\chi_\alpha(x) = \chi_\alpha(kan) := e^{f(an)}\xi_\alpha(k),$$

where $f : AN \to \mathbb{C}$ satisfies
(*i*) $f(1) = 0$,
(*ii*) $\int_{AN} e^{2\Re(f(an))}dadn = 1$ and
(*iii*) $\int_{AN} g(kan)(e^{\overline{f(an)}+f(a_1n_1)})dadn = g(k)$, for $g \in L^2(G)$, $a_1 \in A$, $n_1 \in N$ and the normalized Haar measures da and dn on A and N, respectively.

Proof. Observe that since

$$\chi_\alpha(x) = \chi_\alpha(kan) := e^{f(an)}\xi_\alpha(k),$$

then for any $k \in K$

$$\chi_\alpha(k) = \chi_\alpha(k\cdot 1\cdot 1) := e^{f(1\cdot 1)}\xi_\alpha(k) = \xi_\alpha(k).$$

For any $\alpha_1, \alpha_2 \in A$, we have

$$\langle\chi_{\alpha_1}, \chi_{\alpha_2}\rangle = \int_K (\int_{AN} e^{2\Re(f(an))}dadn)\xi_{\alpha_1}(k)\overline{\xi_{\alpha_2}(k)}dk = \langle\xi_{\alpha_1}, \xi_{\alpha_2}\rangle$$

and

$$\| \chi_\alpha \|_2^2 = \int_K (\int_{AN} e^{2\Re(f(an))}dadn) \mid \xi_{\alpha_1}(k) \mid^2 dk = \| \xi_\alpha \|_2^2 = 1;$$

showing that $\{\chi_\alpha\}_{\alpha\in A}$ is an orthonormal set on G. Its $K-$semicompleteness is also shown as follows. For a pre-assigned $\epsilon > 0$, we have that

$$\| \sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle g, u_{ij}^\lambda\rangle u_{ij}^\lambda - \sum_{j=1}^n \gamma_j \sum_{i=1}^n \beta_{ij} \sum_{\alpha_i^j\in A} \langle g, \chi_{\alpha_i^j}\rangle\chi_{\alpha_i^j} \|_2$$

$$=\| \sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle g, u_{ij}^{\lambda}\rangle u_{ij}^{\lambda} - \int_K [\int_{AN} g(kan)(e^{\overline{f(an)}+f(a_1n_1)})dadn]\cdot$$

$$\sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j\in A} \overline{\xi_{\alpha_i^j}(k)}dk\ \xi_{\alpha_i^j} \|_2$$

$$=\| \sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle g, u_{ij}^{\lambda}\rangle u_{ij}^{\lambda} - \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j\in A} \langle g, \xi_{\alpha_i^j}\rangle \xi_{\alpha_i^j} \|_2 < \epsilon. \ \Box$$

For any $K-$semicomplete orthonormal set $\{\chi_\alpha\}_{\alpha\in A}$ on G the corresponding *Fourier subspace* $L^2(G)(\chi_\alpha)$ of $L^2(G)$ is also given as

$$L^2(G)(\chi_\alpha) := \{f \in L^2(G) : \ f = \sum_{\alpha\in A} \langle f, \chi_\alpha\rangle \chi_\alpha\}$$

while the *prime-Parseval subspace* is

$$L^2(G)'_{\mathfrak{P}}(\chi_\alpha) := \{f \in L^2(G) : \ \langle f, \chi_\alpha\rangle = 0 \text{ (for every } \alpha \in A) \text{ implies } f = 0\}.$$

Clearly $L^2(K)'_{\mathfrak{P}}(\sqrt{d(\lambda)}u_{ij}^{\lambda}) = L^2(K)$ (from Lemma 2.2 (*ii*)), both subspaces $L^2(K)(\chi_\alpha)$ and $L^2(K)'_{\mathfrak{P}}(\chi_\alpha)$ are topologically dense in $L^2(K)$ (from Theorems 3.6 and 3.8 and Corollary 3.9) and there exists an isometry of $L^2(K)'_{\mathfrak{P}}(\chi_\alpha)$ onto $L^2(A)$ (from Theorem 3.13). We shall resume the study of the subspaces $L^2(G)(\chi_\alpha)$ and $L^2(G)'_{\mathfrak{P}}(\chi_\alpha)$ (for connected semisimple Lie groups, G) in another paper.

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SERIES ANALYSIS AND SCHWARTZ ALGEBRAS OF SPHERICAL CONVOLUTIONS ON SEMISIMPLE LIE GROUPS

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Abstract

We give the exact contributions of *Harish-Chandra transform,* $(\mathcal{H}f)(\lambda)$, of Schwartz functions f to the harmonic analysis of *spherical convolutions* and the corresponding L^p- Schwartz algebras on a connected semisimple Lie group G (with finite center). One of our major results gives the proof of how the *Trombi-Varadarajan Theorem* enters into the spherical convolution transform of L^p- Schwartz functions and the generalization of this Theorem under the *full* spherical convolution map.

Subject Classification: $43A85$, $22E30$, $22E46$
Keywords: Harish-Chandra Transforms; Semisimple Lie groups; Harish-Chandra's Schwartz algebras

1 Introduction

Let G be a connected semisimple Lie group with finite center, and denote the Harish-Chandra-type Schwartz spaces of functions on G by $\mathcal{C}^p(G)$, $0 < p \leq 2$. We know that $\mathcal{C}^p(G) \subset L^p(G)$ for every such p, and if K is a maximal compact subgroup of G such that $\mathcal{C}^p(G//K)$ represents the subspace of $\mathcal{C}^p(G)$ consisting of the $K-$bi-invariant functions, Trombi and Varadarajan ([9.]) have shown that the spherical Fourier transform $f \mapsto \widehat{f}$ is a linear topological isomorphism of $\mathcal{C}^p(G//K)$ onto the spaces $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$, $\epsilon = (2/p) - 1$, consisting of rapidly decreasing functions on certain sets $\mathfrak{F}^\epsilon$ of elementary spherical functions.

We show the existence of a *hyper-function* on both G and $\mathfrak{F}^1$ (here named a *spherical convolution*) whose restriction to the group identity element, e, coincides with the *spherical Fourier transforms,* $f \mapsto \widehat{f}$, of Schwartz functions f on G and which affords us the opportunity of embarking on a more inclusive harmonic analysis on G. Indeed [8.] contains a more general Plancherel formula for the collection of these functions. As a function on G its series expansion is in the present paper studied. We show that, aside from the fact that the spherical Fourier transforms, $\widehat{f}(\lambda)$, is the constant term of this series expansion, there is a region in G where the spherical convolution is *essentially* $\widehat{f}(\lambda)$. Various algebras of these functions are thus studied and ultimately embedded in $L^2(G)$. It is however clear that the results in [8.] and in the present paper may be extended to include what may be termed as *the Harish-Chandra-type Schwartz spaces of Eisenstein Integrals on G.*

The following is the breakdown of each of the remaining sections of the paper. §2. contains the preliminaries to the research containing the structure theory, spherical functions and Schwartz algebras on G, while the series analysis of spherical convolutions on G is the subject of §3, where we also extend the *Trombi-Varadarajan Theorem* to all spherical convolutions. The relationship existing among the Schwartz algebras of functions and those of spherical convolutions is considered in §4.

2 Preliminaries

For the connected semisimple Lie group G with finite center, we denote

its Lie algebra by $\mathfrak{g}$ whose *Cartan decomposition* is given as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Denote by θ the *Cartan involution* on $\mathfrak{g}$ whose collection of fixed points is $\mathfrak{k}$. We also denote by K the analytic subgroup of G with Lie algebra $\mathfrak{k}$. K is then a maximal compact subgroup of G. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ with algebraic dual $\mathfrak{a}^*$ and set $A = \exp \mathfrak{a}$. For every $\lambda \in \mathfrak{a}^*$ put

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X, \forall H \in \mathfrak{a}\},$$

and call λ a restricted root of $(\mathfrak{g}, \mathfrak{a})$ whenever $\mathfrak{g}_\lambda \neq \{0\}$.

Denote by $\mathfrak{a}'$ the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$, and call its connected components the *Weyl chambers*. Let $\mathfrak{a}^+$ be one of the Weyl chambers, define the restricted root λ positive whenever it is positive on $\mathfrak{a}^+$ and denote by $\triangle^+$ the set of all restricted positive roots. Members of $\triangle^+$ which form a basis for $\triangle$ and can not be written as a linear combination of other members of $\triangle^+$ are called *simple*. We then have the *Iwasawa decomposition* $G = KAN$, where N is the analytic subgroup of G corresponding to $\mathfrak{n} = \sum_{\lambda \in \triangle^+} \mathfrak{g}_\lambda$, and the *polar decomposition* $G = K \cdot cl(A^+) \cdot K$, with $A^+ = \exp \mathfrak{a}^+$, and $cl(A^+)$ denoting the closure of A^+.

If we set $M = \{k \in K : Ad(k)H = H,\ H \in \mathfrak{a}\}$ and $M' = \{k \in K : Ad(k)\mathfrak{a} \subset \mathfrak{a}\}$ and call them the *centralizer* and *normalizer* of $\mathfrak{a}$ in K, respectively, then (see [5.], p. 284); (i) M and M' are compact and have the same Lie algebra and (ii) the factor $\mathfrak{w} = M'/M$ is a finite group called the *Weyl group*. $\mathfrak{w}$ acts on $\mathfrak{a}^*_{\mathbb{C}}$ as a group of linear transformations by the requirement

$$(s\lambda)(H) = \lambda(s^{-1}H),$$

$H \in \mathfrak{a}$, $s \in \mathfrak{w}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the complexification of $\mathfrak{a}^*$. We then have the *Bruhat decomposition*

$$G = \bigsqcup_{s \in \mathfrak{w}} Bm_sB$$

where $B = MAN$ is a closed subgroup of G and $m_s \in M'$ is the representative of s (i.e., $s = m_sM$). The Weyl group invariant members of a space shall be denoted by the superscript $^{\mathfrak{w}}$ while $|\ \mathfrak{w}\ |$ represents the cardinality of $\mathfrak{w}$.

Some of the most important functions on G are the *spherical functions* which we now discuss as follows. A non-zero continuous function φ on G shall

be called a *(zonial) spherical function* whenever $\varphi(e) = 1$, $\varphi \in C(G//K) := \{g \in C(G)\colon g(k_1 x k_2) = g(x),\, k_1, k_2 \in K,\, x \in G\}$ and $f * \varphi = (f * \varphi)(e) \cdot \varphi$ for every $f \in C_c(G//K)$, where $(f * g)(x) := \int_G f(y) g(y^{-1}x) dy$. This leads to the existence of a homomorphism $\lambda : C_c(G//K) \to \mathbb{C}$ given as $\lambda(f) = (f * \varphi)(e)$. This definition is equivalent to the satisfaction of the functional relation

$$\int_K \varphi(xky) dk = \varphi(x)\varphi(y), \quad x, y \in G.$$

It has been shown by Harish-Chandra [6.] that spherical functions on G can be parametrized by members of $\mathfrak{a}^*_{\mathbb{C}}$. Indeed every spherical function on G is of the form

$$\varphi_\lambda(x) = \int_K e^{(i\lambda - \rho)H(xk)} dk, \ \lambda \in \mathfrak{a}^*_{\mathbb{C}},$$

$\rho = \frac{1}{2} \sum_{\lambda \in \Delta^+} m_\lambda \cdot \lambda$, where $m_\lambda = dim(\mathfrak{g}_\lambda)$, and that $\varphi_\lambda = \varphi_\mu$ iff $\lambda = s\mu$ for some $s \in \mathfrak{w}$. Some of the well-known properties of spherical functions are $\varphi_{-\lambda}(x^{-1}) = \varphi_\lambda(x)$, $\varphi_{-\lambda}(x) = \bar{\varphi}_{\bar{\lambda}}(x)$, $\mid \varphi_\lambda(x) \mid \leq \varphi_{\Re\lambda}(x)$, $\mid \varphi_\lambda(x) \mid \leq \varphi_{i\Im\lambda}(x)$, $\varphi_{-i\rho}(x) = 1$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, while $\mid \varphi_\lambda(x) \mid \leq \varphi_0(x)$, $\lambda \in i\mathfrak{a}^*$, $x \in G$. Also if Ω is the *Casimir operator* on G then

$$\Omega\varphi_\lambda = -(\langle\lambda, \lambda\rangle + \langle\rho, \rho\rangle)\varphi_\lambda,$$

where $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $\langle\lambda, \mu\rangle := tr(adH_\lambda \ adH_\mu)$ for elements H_λ, $H_\mu \in \mathfrak{a}$. This differential equation may be written simply as $\Omega\varphi_\lambda = \gamma(\Omega)(\lambda)\varphi_\lambda$, where $\lambda \mapsto \gamma(\Omega)(\lambda)$ is the well-known *Harish-Chandra homomorphism*. The elements H_λ, $H_\mu \in \mathfrak{a}$ are uniquely defined by the requirement that $\lambda(H) = tr(adH \ adH_\lambda)$ and $\mu(H) = tr(adH \ adH_\mu)$ for every $H \in \mathfrak{a}$ ([5.], Theorem 4.2). Clearly $\Omega\varphi_0 = 0$.

Due to a hint dropped by Dixmier [4.] ($cf.$ [9.]) in his discussion of some functional calculus, it is necessary to recall the notion of a *'positive-definite'* function and then discuss the situation for positive-definite spherical functions. We call a continuous function $f : G \to \mathbb{C}$ (algebraically) positive-definite whenever, for all $x_1, \ldots, x_m$ in G and all $\alpha_1, \ldots, \alpha_m$ in $\mathbb{C}$, we have

$$\sum_{i,j=1}^{m} \alpha_i \bar{\alpha}_j f(x_i^{-1} x_j) \geq 0.$$

It can be shown (*cf.* [5.]) that $f(e) \geq 0$ and $|f(x)| \leq f(e)$ for every $x \in G$ implying that the space $\mathcal{P}$ of all positive-definite spherical functions on G is a subset of the space $\mathfrak{F}^1$ of all bounded spherical functions on G.

We know, by the Helgason-Johnson theorem ([7.]), that

$$\mathfrak{F}^1 = \mathfrak{a}^* + iC_\rho$$

where C_ρ is the convex hull of $\{s\rho : s \in \mathfrak{w}\}$ in $\mathfrak{a}^*$. Defining the *involution* f^* of f as $f^*(x) = \overline{f(x^{-1})}$, it follows that $f = f^*$ for every $f \in \mathcal{P}$, and if $\varphi_\lambda \in \mathcal{P}$, then λ and $\bar{\lambda}$ are Weyl group conjugate, leading to a realization of $\mathcal{P}$ as a subset of $\mathfrak{w} \setminus \mathfrak{a}^*_{\mathbb{C}}$. $\mathcal{P}$ becomes a locally compact Hausdorff space when endowed with the *weak *−topology* as a subset of $L^\infty(G)$.

Let

$$\varphi_0(x) := \int_K \exp(-\rho(H(xk)))dk$$

be denoted as $\Xi(x)$ and define $\sigma : G \to \mathbb{C}$ as

$$\sigma(x) = \|X\|$$

for every $x = k \exp X \in G,\ \ k \in K,\ X \in \mathfrak{a}$, where $\|\cdot\|$ is a norm on the finite-dimensional space $\mathfrak{a}$. These two functions are spherical functions on G and there exist numbers c, d such that

$$1 \leq \Xi(a)e^{\rho(\log a)} \leq c(1 + \sigma(a))^d.$$

Also there exists $r > 0$ such that $c =: \int_G \Xi(x)^2(1+\sigma(x))^r dx < \infty$ ([11.], p. 231). For each $0 \leq p \leq 2$ define $\mathcal{C}^p(G)$ to be the set consisting of functions f in $C^\infty(G)$ for which

$$\mu_{a,b;r}(f) := \sup_G [|f(a;x;b)|\Xi(x)^{-2/p}(1+\sigma(x))^r] < \infty$$

where $a, b \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, the *universal enveloping algebra* of $\mathfrak{g}_{\mathbb{C}}$, $r \in \mathbb{Z}^+, x \in G$, $f(x;b) := \frac{d}{dt}\Big|_{t=0} f(x \cdot (\exp tb))$ and $f(a;x) := \frac{d}{dt}\Big|_{t=0} f((\exp ta) \cdot x)$. We call $\mathcal{C}^p(G)$ the Schwartz space on G for each $0 < p \leq 2$ and note that $\mathcal{C}^2(G)$ is the well-known (see [1.]) Harish-Chandra space of *rapidly decreasing functions* on G. The inclusions

$$C_c^\infty(G) \subset \mathcal{C}^p(G) \subset L^p(G)$$

hold and with dense images. It also follows that $\mathcal{C}^p(G) \subseteq \mathcal{C}^q(G)$ whenever $0 \leq p \leq q \leq 2$. Each $\mathcal{C}^p(G)$ is closed under *involution* and the *convolution*, $*$. Indeed $\mathcal{C}^p(G)$ is a Fréchet algebra ([10.], p. 69). We endow $\mathcal{C}^p(G//K)$ with the relative topology as a subset of $\mathcal{C}^p(G)$.

We shall say a function f on G satisfies a *general strong inequality* if for any $r \geq 0$ there is a constant $c = c_r > 0$ such that

$$| f(y) | \leq c_r \Xi(y^{-1}x)(1 + \sigma(y^{-1}x))^{-r} \quad \forall\, x, y \in G.$$

We observe that if $x = e$ then, using the fact that $\Xi(y^{-1}) = \Xi(y)$ and $\sigma(y^{-1}) = \sigma(y)$, $\forall\, y \in G$, such a function satisfies

$$| f(y) | \leq c_r \Xi(y^{-1})(1 + \sigma(y^{-1}))^{-r} = c_r \Xi(y)(1 + \sigma(y))^{-r},\ \forall\, y \in G,$$

showing that a function on G which satisfies a general strong inequality satisfies in particular a *strong inequality* (in the classical sense of Harish-Chandra, [11.]). Members of $\mathcal{C}^2(G) =: \mathcal{C}(G)$ are those functions f on G for which $f(g_1; \cdot; g_2)$ satisfies the strong inequality, for all $g_1, g_2 \in \mathfrak{U}\,(\mathfrak{g}_{\mathbb{C}})$.We may then define $\mathcal{C}^{(x)}(G)$ to be those functions f on G for which $f(g_1; \cdot; g_2)$ satisfies the general strong inequality, for all $g_1, g_2 \in \mathfrak{U}\,(\mathfrak{g}_{\mathbb{C}})$and a fixed $x \in G$. It is clear that $\mathcal{C}^{(e)}\,(G){=}\,\mathcal{C}(G)$ and that $\bigcup_{x \in G} \mathcal{C}^{(x)}\,(G)$,which contains $\mathcal{C}(G)$, may be given an inductive limit topology. The seminorms defining this topology will be explicitly given in §4.

For any measurable function f on G we define the *spherical Fourier transform* $\widehat{f}$ as

$$\widehat{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx,$$

$\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. It is known (see [3.]) that for $f, g \in L^1(G)$ we have:

(*i.*) $(f * g)^\wedge = \widehat{f} \cdot \widehat{g}$ on $\mathfrak{F}^1$ whenever f (or g) is right - (or left-) K-invariant;

(*ii.*) $(f^*)^\wedge(\varphi) = \overline{\widehat{f}(\varphi^*)}, \varphi \in \mathfrak{F}^1$; hence $(f^*)^\wedge = \overline{\widehat{f}}$ on $\mathcal{P}$: and, if we define $f^{\#}(g) := \int_{K \times K} f(k_1 x k_2) dk_1 dk_2, x \in G$, then

(*iii.*) $(f^{\#})^\wedge = \widehat{f}$ on $\mathfrak{F}^1$.

We shall denote the *spherical Fourier transform* $\widehat{f}(\lambda)$ of $f \in \mathcal{C}(G)$ by $(\mathcal{H}f)(\lambda)$ and refer to it as the *Harish-Chandra transforms* of f. Its major properties are well-known and may be found in [9.]. It should be noted that $(\mathcal{H}f)(\lambda) = \widehat{f}(\lambda) = \int_G f(y)\varphi_{-\lambda}(y)dy = \int_G f(y)\varphi_\lambda(y^{-1})dy = \int_G f(y)\varphi_\lambda(y^{-1}e)dy$ $= (f * \varphi_\lambda)(e)$. That is, the Harish-Chandra transforms of f is the restriction of the function

$$x \mapsto (f * \varphi_\lambda)(x) =: s_{\lambda,f}(x)$$

on G to the identity element. It is therefore worthwhile to explore $s_{\lambda,f}(x)$ in some details for all $x \in G$ in order to put its behaviour at $x = e$ (as the Harish-Chandra transforms of f) in a proper and larger perspective.

The beauty of studying the entirety of the function $s_{\lambda,f}(x)$, for $\lambda \in \mathfrak{a}^*_\mathbb{C}$, $f \in \mathcal{C}^p(G)$, $x \in G$, which we shall explore in this paper, is that it could be viewed as a transformation in six (6) different ways; As

$$\text{(1.)} \qquad x \mapsto k_1(\lambda) := s_{\lambda,f}(x), \text{ for any } f \in \mathcal{C}^p(G)$$

and

$$\text{(2.)} \qquad x \mapsto k_2(f) := s_{\lambda,f}(x), \text{ for any } \lambda \in \mathfrak{a}^*_\mathbb{C},$$

(from where the Plancherel formula for the space of functions $x \mapsto k_2(f)$ has recently been computed in [8.]) both of which are maps on G; or as

$$\text{(3.)} \qquad f \mapsto l_1(\lambda) := s_{\lambda,f}(x), \text{ for any } x \in G$$

(which, at $x = e$, led Harish-Chandra to the consideration of $f \mapsto (\mathcal{H}f)(\lambda)$: *cf*. [9.]) and

$$\text{(4.)} \qquad f \mapsto l_2(x) := s_{\lambda,f}(x), \text{ for any } \lambda \in \mathfrak{a}^*_\mathbb{C},$$

both of which are maps on $\mathcal{C}^p(G)$; or as

$$\text{(5.)} \qquad \lambda \mapsto m_1(f) := s_{\lambda,f}(x), \text{ for any } x \in G$$

and

$$\text{(6.)} \qquad \lambda \mapsto m_2(x) := s_{\lambda,f}(x), \text{ for any } f \in \mathcal{C}^p(G),$$

both of which are maps on $\mathfrak{a}^*_\mathbb{C}$. Hence the function $x \mapsto s_{\lambda,f}(x)$ may rightly be called an *hyper-function* on G whose major contribution to harmonic analysis would be to *absorb* other known functions of the subject and put their results in *proper perspectives,* as we shall establish here for the *Harish-Chandra*

transform and *Trombi-Varadarajan Theorem.*

In order to know the image of the spherical Fourier transform when restricted to $\mathcal{C}^p(G//K)$ we need the following spaces that are central to the statement of the well-known result of Trombi and Varadarajan [9.]. Let C_ρ be the closed convex hull of the (finite) set $\{s\rho : s \in \mathfrak{w}\}$ in $\mathfrak{a}^*$, i.e.,

$$C_\rho = \left\{ \sum_{i=1}^{n} \lambda_i (s_i \rho) : \lambda_i \geq 0, \ \ \sum_{i=1}^{n} \lambda_i = 1, \ \ s_i \in \mathfrak{w} \right\}$$

where we recall that, for every $H \in \mathfrak{a}$, $(s\rho)(H) = \frac{1}{2} \sum_{\lambda \in \Delta^+} m_\lambda \cdot \lambda(s^{-1}H)$.

Now for each $\epsilon > 0$ set $\mathfrak{F}^\epsilon = \mathfrak{a}^* + i\epsilon C_\rho$. Each $\mathfrak{F}^\epsilon$ is convex in $\mathfrak{a}^*_{\mathbb{C}}$ and

$$int(\mathfrak{F}^\epsilon) = \bigcup_{0<\epsilon'<\epsilon} \mathfrak{F}^{\epsilon'}$$

([9.], Lemma (3.2.2)). Let us define $\mathcal{Z}(\mathfrak{F}^0) = \mathcal{S}(\mathfrak{a}^*)$ and, for each $\epsilon > 0$, let $\mathcal{Z}(\mathfrak{F}^\epsilon)$ be the space of all $\mathbb{C}$-valued functions Φ such that $(i.)$ Φ is defined and holomorphic on $int(\mathfrak{F}^\epsilon)$, and $(ii.)$ for each holomorphic differential operator D with polynomial coefficients we have $\sup_{int(\mathfrak{F}^\epsilon)} |D\Phi| < \infty$.

The space $\mathcal{Z}(\mathfrak{F}^\epsilon)$ is converted to a Fréchet algebra by equipping it with the topology generated by the collection, $\| \cdot \|_{\mathcal{Z}(\mathfrak{F}^\epsilon)}$, of seminorms given by $\|\Phi\|_{\mathcal{Z}(\mathfrak{F}^\epsilon)} := \sup_{int(\mathfrak{F}^\epsilon)} |D\Phi|$. It is known that $D\Phi$ above extends to a continuous function on all of $\mathfrak{F}^\epsilon$ ([9.], pp. $278 - 279$). An appropriate subalgebra of $\mathcal{Z}(\mathfrak{F}^\epsilon)$ for our purpose is the closed subalgebra $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$ consisting of $\mathfrak{w}$-invariant elements of $\mathcal{Z}(\mathfrak{F}^\epsilon)$, $\epsilon \geq 0$. The following (known as the *Trombi-Varadarajan Theorem*) is the major result of [9.] : *Let* $0 < p \leq 2$ *and set* $\epsilon = (2/p) - 1$. *Then the spherical Fourier transform* $f \mapsto \widehat{f}$ *is a linear topological algebra isomorphism of* $\mathcal{C}^p(G//K)$ *onto* $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$. That is, the topological algebra $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$ is an isomorphic copy or a realization of $\mathcal{C}^p(G//K)$.

In order to find other isomorphic copies or realizations of $\mathcal{C}^p(G//K)$ under the more inclusive general transformation map

$$f \mapsto l_1(\lambda) := s_{\lambda,f}(x), \text{ for any } x \in G,$$

we shall now introduce a more general algebra, $\bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$, of $\mathbb{C}-$valued functions on $int(\mathfrak{F}^\epsilon) \times G$ which, when restricted to $int(\mathfrak{F}^\epsilon) \times \exp(N_0)$, coincides

with $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$. The form of this new algebra is suggested by Theorem 3.5. Set $\mathcal{Z}_G(\mathfrak{F}^0) = \mathcal{S}(\mathfrak{a}^*) \times G$ and let $\mathcal{Z}_G(\mathfrak{F}^\epsilon)$, $\epsilon > 0$, be the collection of all $\mathbb{C}$−valued functions Ψ $((\lambda, x) \mapsto \Psi(\lambda, x),\ \forall\ (\lambda, x) \in int(\mathfrak{F}^\epsilon) \times G)$ such that

($i.$) Ψ is holomorphic in the variable λ, analytic in x and spherical on G;

($ii.$) $\sup_{int(\mathfrak{F}^\epsilon)} |D_1\Psi| < \infty$ and $\sup_G |\Psi D_2| < \infty$, for every holomorphic differential operator D_1 with polynomial coefficients and every left-invariant differential operator D_2 on G and

($iii.$) the restriction of Ψ to $int(\mathfrak{F}^\epsilon) \times \{e\}$ (or to $int(\mathfrak{F}^\epsilon) \times \exp(N_0(A^+))$, for some zero neighbourhood $N_0(A^+)$ in $\mathfrak{g}$, as will later be seen in Theorem 3.5) is (a non-zero constant multiple of) the Harish-Chandra transform, $(\mathcal{H}f)(\lambda) = \hat{f}$.

It may be shown, in exact manner as for $\mathcal{Z}(\mathfrak{F}^\epsilon)$ above, that the space $\mathcal{Z}_G(\mathfrak{F}^\epsilon)$ is converted to a Fréchet algebra by equipping it with the topology generated by the collection, $\|\cdot\|_{\mathcal{Z}_G(\mathfrak{F}^\epsilon)}$, of seminorms given by

$$\|\Psi\|_{\mathcal{Z}_G(\mathfrak{F}^\epsilon)} := \sup_{int(\mathfrak{F}^\epsilon)\times G} |D_1 \Psi D_2|.$$

An appropriate subalgebra of $\mathcal{Z}_G(\mathfrak{F}^\epsilon)$ for our purpose is the closed subalgebra $\bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$ consisting of $\mathfrak{w}$-invariant elements of $\mathcal{Z}_G(\mathfrak{F}^\epsilon)$, $\epsilon \geq 0$. By the time Theorem 3.5 is established it will be clear that $\bar{\mathcal{Z}}_{\{x\}}(\mathfrak{F}^\epsilon) \simeq \bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$, for every x in some zero neighbourhood $N_0(A^+)$ in $\mathfrak{g}$. In particular, $\bar{\mathcal{Z}}_{\{e\}}(\mathfrak{F}^\epsilon) \simeq \bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$.

3 Series Analysis of Spherical Convolutions

Let $f \in \mathcal{C}(G)$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, we recall from [8.] the definition of *spherical convolutions,* $s_{\lambda,f}$, on G corresponding to the pair (λ, f) as

$$s_{\lambda,f}(x) := (f * \varphi_\lambda)(x),\ \ x \in G.$$

We already know that $s_{\lambda,f}(e) = (\mathcal{H}f)(\lambda)$, where e is the identity element of G and $\lambda \in i\mathfrak{a}^*$. This relation between a function on G at the identity element and another function on $i\mathfrak{a}^*$ suggests we study the full contribution of the Harish-Chandra transforms, $(\mathcal{H}f)(\lambda)$, of f to the properties of $x \mapsto s_{\lambda,f}(x)$

and to seek other functions on $i\mathfrak{a}^*$ which have not been known in the harmonic analysis of G, but still contribute to a deeper understanding of the structure of G.

In order to explore the nature of this idea we consider opening up the spherical convolutions $x \mapsto s_{\lambda,f}(x)$ via its *Taylor's series expansion.*

Lemma 3.1. *Let N_0 be a neighbourhood of origin in $\mathfrak{g}$ and t be sufficiently small in $\mathbb{R}$ (say $0 \le t \le 1$). Then*

$$s_{\lambda,f}(x \exp tX) = \sum_{n=0}^{\infty} \frac{t^n}{n!}[\tilde{X}^n s_{\lambda,f}](x),$$

where for every $X \in N_0$ we set $[\tilde{X}^n s_{\lambda,f}](x) = \frac{d^n}{du^n} s_{\lambda,f}(x \exp uX)_{|u=0}$

Proof. The proof follows from a direct application of *Taylor's series expansion,* [5.], *p.* 105. $\square$

At $x = e$ and $t = 1$ the formula in the Lemma becomes

$$s_{\lambda,f}(\exp X) = \sum_{n=0}^{\infty} \frac{1}{n!}[\tilde{X}^n s_{\lambda,f}](e) = s_{\lambda,f}(e) + \sum_{n=1}^{\infty} \frac{1}{n!}[\tilde{X}^n s_{\lambda,f}](e)$$

$$= (\mathcal{H}f)(\lambda) + \sum_{n=1}^{\infty} \frac{1}{n!}[\tilde{X}^n s_{\lambda,f}](e), \quad X \in N_0.$$

This observation leads quickly to the following result which gives the exact contribution of the Harish-Chandra transforms to the study of spherical convolutions.

Lemma 3.2. *The Harish-Chandra transforms, $\lambda \mapsto (\mathcal{H}f)(\lambda)$, $f \in \mathcal{C}(G)$, is the constant term in the (Taylor's) series expansion of spherical convolutions, $x \mapsto s_{\lambda,f}(x)$ around $x = e$, for every $\lambda \in \mathfrak{a}^*$.* $\square$

It may be deduced, from the expansion leading to the proof Lemma 3.2, that the only time the remaining terms in $s_{\lambda,f}(\exp X)$, after the (*non-zero*) constant term $(\mathcal{H}f)(\lambda)$, could vanish is when the differential operator $\tilde{X} = 0$. That is, when $X = 0$. It therefore follows that the well-known (Harish-Chandra) harmonic analysis on G ([1.], [2.], [9.] and [11.]) has always been

that of the consideration of the map $X \mapsto s_{\lambda,f}(\exp X)$ at only $X = 0$, which is the origin of $\mathfrak{g}$ or which corresponds to the identity point of $\exp(\mathfrak{g})$. Hence, since the constant term, $(\mathcal{H}f)(\lambda)$, of $s_{\lambda,f}(\exp X)$ corresponds indeed to the consideration of the constant term in the asymptotic expansion of (zonal) spherical functions, φ_λ, it also follows that other terms in the expansion of φ_λ may be needed to completely understand $f \mapsto s_{\lambda,f}(x)$.

The expression for $s_{\lambda,f}(\exp X)$ therefore suggests that a *full* harmonic analysis of G may be attained from a close study of the remaining contributions of the *transform* of f given as

$$\lambda \longmapsto \frac{t^n}{n!}[\tilde{X}^n s_{\lambda,f}](x),$$

for all $X \in N_0$, $n \in \mathbb{N} \cup \{0\}$, $x \in G$, $f \in \mathcal{C}(G)$ and sufficiently small values of t, in the same manner that its constant term,

$$\lambda \longmapsto (\mathcal{H}f)(\lambda)$$

had been considered.

However before considering the transformational properties of spherical convolutions we note the following lemmas which lead to a more inclusive view of the Trombi-Varadarajan Theorem and prepares the ground for its generalization.

Lemma 3.3. *Let N_0 be a neighbourhood of origin in $\mathfrak{g}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and t be sufficiently small in $\mathbb{R}$ (say $0 \leq t \leq 1$). Then*

$$s_{\lambda,f}(x \exp tX) = [\sum_{n=0}^{\infty} \frac{t^n}{n!} \gamma(\frac{d^n}{du^n})(\lambda)_{|u=0}] \cdot s_{\lambda,f}(x),$$

for every $X \in N_0$, $x \in G$, $f \in \mathcal{C}(G)$.

Proof. We note here that

$$[\tilde{X} s_{\lambda,f}](x) = \frac{d}{du} s_{\lambda,f}(x \exp uX)_{|u=0} = \frac{d}{du}(f * \varphi_\lambda)(x \exp uX)_{|u=0}$$

$$= (f * \frac{d}{du}\varphi_\lambda)(x \exp uX)_{|u=0} = \gamma(\frac{d}{du})(\lambda) \cdot (f * \varphi_\lambda)(x \exp uX)_{|u=0}.$$

Hence

$$[\tilde{X}^n s_{\lambda,f}](x) = \gamma(\frac{d^n}{du^n})(\lambda)_{|u=0}\cdot(f*\varphi_\lambda)(x\exp uX)_{|u=0} = \gamma(\frac{d^n}{du^n})(\lambda)_{|u=0}\cdot s_{\lambda,f}(x). \ \square$$

The particular case of setting $x = e$ and $t = 1$ in Lemma 3.3 introduces the Harish-Chandra transforms, $(\mathcal{H}f)(\lambda)$, into the analysis of this series, proving the following.

Lemma 3.4. *Let N_0 be a neighbourhood of origin in $\mathfrak{g}$, $f \in \mathcal{C}(G)$ and $\lambda \in \mathfrak{a}^*$. Then the spherical convolution function, $x \mapsto s_{\lambda,f}(x)$ is a non-zero constant multiple of the Harish-Chandra transforms, $(\mathcal{H}f)(\lambda)$, on* $\exp(N_0)$.

Proof. Set $x = e$ and $t = 1$ into Lemma 3.3 to have

$$s_{\lambda,f}(\exp X) = [\sum_{n=0}^{\infty}\frac{1}{n!}\gamma(\frac{d^n}{du^n})(\lambda)_{|u=0}]\cdot s_{\lambda,f}(e) = [\sum_{n=0}^{\infty}\frac{1}{n!}\gamma(\frac{d^n}{du^n})(\lambda)_{|u=0}]\cdot(\mathcal{H}f)(\lambda),$$

with $\sum_{n=0}^{\infty}\frac{1}{n!}\gamma(\frac{d^n}{du^n})(\lambda)_{|u=0} = 1 + [\sum_{n=1}^{\infty}\frac{1}{n!}\gamma(\frac{d^n}{du^n})(\lambda)_{|u=0}] \neq 0.$ $\square$

Let us denote the non-zero constant in Lemma 3.4 above by κ. The following theorem is a consequence of normalizing the spherical convolutions in Lemma 3.4.

Theorem 3.5. (Trombi-Varadarajan Theorem for Spherical Convolutions) *Let $0 < p \leq 2$, set $\epsilon = (2/p) - 1$ and $x \in$ $\exp(N_0)$. Set $\widehat{f}_x(\lambda) = \frac{1}{\kappa}s_{\lambda,f}(x)$ for $f \in \mathcal{C}^p(G//K)$. Then the spherical convolution transforms $f \mapsto \widehat{f}_x$ is a linear topological algebra isomorphism of $\mathcal{C}^p(G//K)$ onto $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$.* $\square$

We recover the Trombi-Varadarajan Theorem for Harish-Chandra transforms by setting $x = e$ in Theorem 3.5. Indeed, Theorem 3.5 above says that every $x \in \exp(N_0)$ (and not just $x = e$) gives a topological algebra isomorphism between $\mathcal{C}^p(G//K)$ and $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$. However if $x \in G\setminus\exp(N_0)$, for any neighborhood N_0 of zero in $\mathfrak{g}$, Trombi-Varadarajan Theorem may not be appropriate and it may be necessary to seek a more general realization of $\mathcal{C}^p(G//K)$ under the map $f \mapsto l_1(\lambda) := s_{\lambda,f}(x)$, for any $x \in G$. Before considering another major result of this paper, giving the fine structure of spherical convolution functions, we state a result on the finiteness of a central integral usually used in the estimation of many other integrals of harmonic

analysis on semisimple Lie groups.

To this end we define, for every $x \in G$, the function $x \mapsto d(x)$ as

$$d(x) = \int_G \Xi^2(y^{-1}x)(1+\sigma(y^{-1}x))^{-r}dy.$$

We observe here that

$$d(e) = \int_G \Xi^2(y^{-1})(1+\sigma(y^{-1}))^{-r}dy = \int_G \Xi^2(y)(1+\sigma(y))^{-r}dy,$$

which is a constant whose proof of finiteness may be found in [11.], *p.* 231. This constant is crucial to all harmonic analysis of $\mathcal{C}(G)$ and, in particular, to the embedding of $\mathcal{C}(G)$ in $L^2(G)$. It is therefore important to understand the nature of $d(x)$ for all $x \in G$ in order to employ it in a more inclusive harmonic analysis on G. We consider the nature of this integral in the following.

Lemma 3.6. *Let $x \in G$. Then there exist $r \geq 0$ such that*

$$d(x) = \int_G \Xi^2(y^{-1}x)(1+\sigma(y^{-1}x))^{-r}dy < \infty.$$

Proof. We already know that $\Xi(y^{-1}x) \leq 1$. Also

$$1+\sigma(y^{-1}x) \leq (1+\sigma(y^{-1}))(1+\sigma(x)) = (1+\sigma(y))(1+\sigma(x)).$$

It follows therefore that

$$d(x) \leq \int_G (1+\sigma(y^{-1}x))^{-r}dy \leq (1+\sigma(x)) \int_G (1+\sigma(y))dy.$$

The last integral in the above inequality is finite if we embark on its computation via the polar decomposition, $G = K \cdot cl(A^+) \cdot K$, of G. □

Theorem 3.7. *Let N_0 be a neighbourhood of origin in $\mathfrak{g}$ where f is a measurable function on G which satisfies the general strong inequality. The integral defining the spherical convolution function, $x \mapsto s_{\lambda,f}(x)$, is absolutely and uniformly convergent for all $x \in \exp(N_0)$, $\lambda \in i\mathfrak{a}^*$. Moreover the transforms $\lambda \mapsto s_{\lambda,f}(x)$ of f, with $x \in \exp(N_0)$, is a continuous function on $i\mathfrak{a}^*$. If $r \geq 0$ is such that $d(x) = \int_G \Xi^2(y^{-1}x)(1+\sigma(y^{-1}x))^{-r}dy < \infty$, $x \in G$, then*

$$\mid s_{\lambda,f}(x) \mid \leq d(x) \cdot \mu_{1,1,r}(f), \quad x \in G, \ \lambda \in i\mathfrak{a}^*.$$

Proof. We recall that $|\ \varphi_\lambda(x)\ | \leq \varphi_0(x) = \Xi(x),\ x \in G,\ \lambda \in i\mathfrak{a}^*$. Hence

$$|\ (f*\varphi_\lambda)(x)\ | \leq \int_G |\ f(y)\varphi_\lambda(y^{-1}x)\ |\ dy \leq \mu_{1,1,r}(f) \int_G \Xi^2(y^{-1}x)(1+\sigma(y^{-1}x))^{-r}dy$$

$= d(x) \cdot \mu_{1,1,r}(f)$. Continuity follows from the use of the Lebesgue's dominated convergence theorem. □

The following well-known result on the foundational properties of the Harish-Chandra transforms, $\lambda \mapsto (\mathcal{H}f)(\lambda),\ \lambda \in i\mathfrak{a}^*$, now follows from the general outlook given by Theorem 3.7.

Corollary 3.8. ([9.]) *Let f be a measurable function on G which satisfies the strong inequality. The integral defining the Harish-Chandra transforms,*

$$(\mathcal{H}f)(\lambda) = \int_G f(x)\varphi_\lambda(x)dx,$$

is absolutely and uniformly convergent for all $\lambda \in i\mathfrak{a}^*$ and is continuous on $i\mathfrak{a}^*$. If $r \geq 0$ is such that $d = \int_G \Xi^2(y)(1+\sigma(y))^{-r}dy < \infty$, then

$$(\mathcal{H}f)(\lambda)\ | \leq d\mu_{1,1,r}(f),\quad \lambda \in i\mathfrak{a}^*.$$

Proof. Set $X = 0$ in Theorem 3.7 to have the first results. The inequality follows if we set $x = e$ and observe that $d(e) = \int_G \Xi^2(y^{-1})(1+\sigma(y^{-1}))^{-r}dy = d$. □

We now consider the image of $\mathcal{C}^p(G//K)$ under the *full* spherical convolution map, $f \mapsto l_1(\lambda) := s_{\lambda,f}(x)$, for any $x \in G$. In order to discuss this we have two options. One of the options is to introduce wave-packet that will still have its domain as $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$ while using an appropriate Plancherel measure on $\mathfrak{F}^\epsilon$. This option has been explored in [8.], p. 34, where the L^2 Plancherel measure, $d\zeta_{x,\lambda}$ on $\mathfrak{F}^1$ for the spherical convolution function (when viewed as a function on G) was defined to absorb the group variable, x. The results therein suggest that the image of $\mathcal{C}^p(G//K)$ under the *full* spherical convolution map is indeed possible.

The second option is to retain the *spherical Bochner measure,* $d\lambda$, on (a subset of) $\mathfrak{F}^{\epsilon}$ and define the wave-packet as a map on the Fréchet algebra $\bar{\mathcal{Z}}_G(\mathfrak{F}^{\epsilon})$. This will reflect the nature of the full spherical convolution map as a transform of members of $\mathcal{C}^p(G//K)$ whose arguments are (generally) taken from $int(\mathfrak{F}^{\epsilon}) \times G$ (and not just from $int(\mathfrak{F}^{\epsilon})$ as in the first option). This is the option we shall explore in the present paper.

To this end recall the Fréchet algebra $\bar{\mathcal{Z}}_G(\mathfrak{F}^{\epsilon})$, $\forall\ \epsilon > 0$, let $\Psi \in \bar{\mathcal{Z}}_G(\mathfrak{F}^{\epsilon})$ and set

$$N_0(A^+) = N_0 \cap A^+,$$

where N_0 is a zero neighbourhood in $\mathfrak{g}$. It is clear that $N_0(A^+)$ is also a zero neighbourhood in $\mathfrak{g}$ and that $\Psi = \Psi(\lambda, x)$, for all $(\lambda, x) \in int(\mathfrak{F}^{\epsilon}) \times G$. It follows, from Theorem 3.5, that $\bar{\mathcal{Z}}_{\{x\}}(\mathfrak{F}^{\epsilon}) \simeq \bar{\mathcal{Z}}(\mathfrak{F}^{\epsilon})$, for every $x \in \exp(N_0(A^+))$. We then have the following.

Lemma 3.9. *For every* $x \in \exp(N_0(A^+))$ *and* $\Psi \in \bar{\mathcal{Z}}_G(\mathfrak{F}^{\epsilon})$, *we have that* $\Psi(\lambda, x) = \Phi(\lambda)$, *for some* $\Phi \in \bar{\mathcal{Z}}(\mathfrak{F}^{\epsilon})$.

We now employ these remarks to define a map from $\bar{\mathcal{Z}}_G(\mathfrak{F}^{\epsilon})$ to $\mathcal{C}^p(G//K)$ as follows. Let $a \in \bar{\mathcal{Z}}_G(\mathfrak{F}^{\epsilon})$ and $\lambda \mapsto c(\lambda)$ be the Harish-Chandra $c-$function defined on $\mathfrak{F}_I := i\mathfrak{a}^*$. We associate to every $a \in \bar{\mathcal{Z}}_G(\mathfrak{F}^{\epsilon})$ the function φ_a on G defined as

$$\varphi_a(x) =\mid \mathfrak{w} \mid^{-1} \int_{\mathfrak{F}_I} a(-\lambda, x)\varphi_{-\lambda}(x)c(-\lambda)^{-1}c(\lambda)^{-1}d\lambda, \quad x \in G.$$

It should be noted here that

$$\begin{aligned}\varphi_a(x) &=\mid \mathfrak{w} \mid^{-1} \int_{\mathfrak{F}_I} a(-\lambda, x)\varphi_{-\lambda}(x)c(-\lambda)^{-1}c(\lambda)^{-1}d\lambda \\ &=\mid \mathfrak{w} \mid^{-1} \int_{\mathfrak{F}_I} a(\lambda, x)\varphi_{\lambda}(x)c(\lambda)^{-1}c(-\lambda)^{-1}d(-\lambda) \\ &=\mid \mathfrak{w} \mid^{-1} \int_{\mathfrak{F}_I} a(\lambda, x)\varphi_{\lambda}(x)c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda,\end{aligned}$$

which is due to the invarianve of $d\lambda$, and that

$$\varphi_a(k_1xk_2) = \varphi_a(x),$$

$\forall\, x \in G,\ k_1, k_2 \in K$, being a property inherited from a and φ_λ.

The (extra) requirement of being spherical on G placed on members of $\bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$ may at first be seen as a restriction, when compared to the requirements on members of $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$. It however turns out that this extra requirement is what is needed to assure us of the generalization of the *classical* wave-packets (of Trombi-Varadarajan) on G to all of $x \mapsto \varphi_a(x)$. This is established as follows.

Lemma 3.10. *Let $a \in \bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$ and $N_0(A^+)$ be as defined above. Then, for every $x \in \exp(N_0(A^+))$, the map $x \mapsto \varphi_a(x)$ is the classical wave-packet of G.*

Proof. We observe that, with $\exp tH \in \exp(N_0(A^+))$,

$$a(\lambda, x) = a(\lambda, k_1 \exp tH k_2) = a(\lambda, \exp tH) = \Phi(\lambda),$$

for some $\Phi \in \bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$. Here we have employed the spherical property of a on G in the second equality and Lemma 3.9 in the third equality. $\square$

The above Lemma shows that the definition and properties of the map $x \mapsto \varphi_a(x), x \in G$, is consistent with the relationship (in Lemma 3.4) existing between spherical convolutions, $s_{\lambda,f}(x)$ and the Harish-Chandra transfroms, $(\mathcal{H}f)(\lambda)$. Hence in order to extend Trombi-Varadarajan Theorem (which gives the image of the algebra $\mathcal{C}^p(G//K)$ under $f \mapsto (\mathcal{H}f)(\lambda)$) to all $x \in G$ (under the spherical convolution tranform), it will be necessary to show that $x \mapsto \varphi_a(x)$ is the wave-packet of $f \mapsto s_{\lambda,f}(x)$ for all $x \in G$. According to Lemma 3.10, this needs only be done for those $x = k_1 \exp tH k_2$ in G with $\exp tH \notin \exp(N_0(A^+))$, for any neighbourhood, N_0, of zero in $\mathfrak{g}$. We however give a self-contained discussion of these results, the first of which is given below.

Theorem 3.11. *$\varphi_a \in \mathcal{C}^p(G//K)$ for every $a \in \bar{\mathcal{Z}}_G(\mathfrak{F}^\epsilon)$.*

In order to establish this Theorem we prove some lemmas which give appropriate background for it. Indeed we derive an appropriate bound for $\mid \varphi_a(h;u) \mid$, where $u \in \mathfrak{U}(\mathfrak{g}_\mathbb{C})$ and h is well-chosen, and the appropriate collection of seminorms are also in place.

4 Algebras of Spherical Convolutions

We now consider the various algebras of spherical convolutions that have emanated in the course of this research and their relationship with the Harish-Chandra Schwartz algebra, $\mathcal{C}(G)$, on G as well as its distinguished commutative subalgebra, $\mathcal{C}(G//K)$, of (elementary) spherical functions.

Define $\mathcal{C}_\lambda(G) = \{s_{\lambda,f} : f \in \mathcal{C}(G)\}$ and set $\mathcal{C}_{\lambda,0}(G) = \{s_{\lambda,\varphi_\lambda}\}$, for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. It is clear that $\bigcup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}} \mathcal{C}_\lambda(G)$ is contained in $\mathcal{C}(G)$. We may therefore topologize $\bigcup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}} \mathcal{C}_\lambda(G)$ by giving it the *relative topology* from the topology defined on $\mathcal{C}(G)$ by the seminorms, $\mu_{a,b,r}$.

Lemma 4.1. *The inclusions*

$$[\bigcup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}} \mathcal{C}_{\lambda,0}(G)] \subset \mathcal{C}(G//K) \subset [\bigcup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}} \mathcal{C}_\lambda(G)] \subset \mathcal{C}(G)$$

are all proper. □

Theorem 4.2. $\bigcup_{\lambda\in\mathfrak{F}^1} \mathcal{C}_\lambda(G)$ *is a closed subalgebra of* $\mathcal{C}(G)$.
Proof. We recall that $\mu_{a,b;r}(f * \varphi_\lambda) \leq c\mu_{1,b;r+r_0}(f) \cdot \mu_{a,1;r}(\varphi_\lambda)$, where $c := \int_G \Xi^2(x)(1+\sigma(x))^{-r_0}dx < \infty$ for some $r_0 \geq 0$. However

$$\begin{aligned}\mu_{a,1;r}(\varphi_\lambda) &= \sup_G[|\varphi_\lambda(1;x;a)| \cdot \Xi(x)^{-1}(1+\sigma(x))^r] \\ &=| \gamma(a)(\lambda) | \cdot \sup_G[|\varphi_\lambda(x)| \cdot \Xi(x)^{-1}(1+\sigma(x))^r] \\ &\leq M \mid \gamma(a)(\lambda) \mid \cdot \sup_G[\Xi(x)^{-1}(1+\sigma(x))^r] < \infty\end{aligned}$$

(since φ_λ is bounded for all $\lambda \in \mathfrak{F}^1$).

Hence $\mu_{a,b;r}(f * \varphi_\lambda) < \infty$, $\forall\, \lambda \in \mathfrak{F}^1$. □

It may be recalled that members of $\mathcal{C}(G)$ are exactly those functions on G whose left and right derivatives satisfy the *strong inequality*. In the light of this observation we define $\mathcal{C}^{(x)}(G)$ as exactly those functions on G whose

left and right derivatives satisfy the *general strong inequality,* for each $x \in G$. Explicitly we set $\mathcal{C}^{(x)}(G)$ as

$$\mathcal{C}^{(x)}(G) = \{f : G \mapsto \mathbb{C} : \sup_{y \in G}[|f(a; y; b)| \cdot \Xi(y^{-1}x)^{-1}(1 + \sigma(y^{-1}x))^r] < \infty\},$$

$x \in G$. A collection of seminorms on each of $\mathcal{C}^{(x)}(G)$ may be given by

$$\mu^{(x)}_{a,b;r}(f) := \sup_{y \in G}[|f(a; y; b)| \cdot \Xi(y^{-1}x)^{-1}(1 + \sigma(y^{-1}x))^r].$$

It is however clear that $\mathcal{C}^{(e)}(G) = \mathcal{C}(G)$, so that $\mathcal{C}(G) \subset \bigcup_{x \in G} \mathcal{C}^{(x)}(G)$.

Theorem 4.3. *The natural inclusion* $\bigcup_{x \in G} \mathcal{C}^{(x)}(G) \subset L^2(G)$ *has a dense image.*

Proof. It is known that the natural inclusion of $\mathcal{C}(G)$ in $L^2(G)$ has a dense image, [1.]. The result therefore follows if we recall that, as sets of functions,

$$\mathcal{C}(G) \subset \bigcup_{x \in G} \mathcal{C}^{(x)}(G) \subset L^2(G),$$

where the second inclusion holds from the fact that $d(x) < \infty$, $x \in G$. $\square$

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QUANTUM RELATIVISTIC EQUATION FOR A PROBABILTY AMPLITUDE

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Abstract

The relativistic quantum equation is proposed for the complex wave function, which has the meaning of a probability amplitude. The Lagrangian formulation of the proposed theory is developed. The problem of spreading of a wave packet in an unlimited space is solved. The relativistic correction to the energy levels of a harmonic oscillator is found, leading to a violation of their equidistance.

Key words: Schrödinger equation, quantum mechanics, relativistic equation, wave function, probability amplitude, harmonic oscillator

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1 Introduction

Fundamental issues related to the completeness of the description of reality in quantum mechanics were discussed long ago in well-known works [1,2] and then in many others. These discussions continue in present time. The Schrödinger equation for the complex wave function ψ, which, according to Born's interpretation, has the meaning of a probability amplitude, is nonrelativistic. This means that the theory allows propagation of signals at arbitrarily high speeds. Perhaps, some of the discussed difficulties and paradoxes of quantum mechanics are related to this circumstance. It should also be noted that quantum theory is used in the description and interpretation of experiments with photons, which are relativistic objects. So far there is no generally accepted relativistic equation for the field, which could be interpreted as a probability amplitude and would allow a Born probabilistic interpretation. The probabilistic interpretation of the complex field within the framework of Dirac theory was discussed in works [3,4].

In this paper the relativistic equation is proposed for the complex scalar field, allowing its physical interpretation as a probability amplitude. The theory is formulated within the framework of the Lagrangian formalism. The spreading of a wave packet in an unlimited space is considered. The relativistic correction has been found in the theory of a harmonic oscillator, leading to a violation of the equidistance of levels. The issue of the completeness of the quantum description is not addressed in the paper.

2 Relativistic equation for the complex scalar field

The recipe for the transition from the classical to the quantum mechanical equation consists, as is known [5], in replacing momentum and energy in the classical Hamiltonian $H = \mathbf{p}^2/2m$ with the operators $\mathbf{p} \to -i\hbar\nabla$ and $H \to i\hbar\,\partial/\partial t$, which act on the complex function ψ. In the relativistic case, the relationship between energy and momentum is given by formula $H = \sqrt{m^2c^4 + c^2\mathbf{p}^2}$, where m is the particle mass, c is the speed of light. Using the indicated substitution, we arrive at the relativistic equation

$$i\hbar\frac{\partial\psi}{\partial t} = \sigma\sqrt{m^2c^4 - \hbar^2c^2\Delta}\;\psi, \tag{1}$$

where Δ is the Laplace operator, $\sigma = \pm 1$. This equation is inconvenient for a number of reasons. Firstly, the time and space coordinates enter here unequally, so that it does not have an explicitly covariant form. Secondly, because in Eq. (1) the Laplace operator appears under the sign of the root, and, therefore, this equation is nonlocal. In order to get rid of the noted formal shortcomings, the transition to the quantum mechanical equation is usually carried out in the expression for the square of energy $H^2 = m^2c^4 + c^2\mathbf{p}^2$, which leads to the Klein-Gordon-Fock equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \lambda\!\!\!^{-}{}^{-2}\right)\psi = 0, \tag{2}$$

where $\lambda\!\!\!^{-} \equiv \hbar/mc$ is the Compton length. In the following we will also use 4-dimensional notation, in which an arbitrary 4-vector is $B \equiv B_\mu \equiv (\mathbf{B}, B_4 = iB_0)$, in particular $x \equiv x_\mu \equiv (\mathbf{x}, x_4 = ix_0 = ict)$. The scalar product of two vectors is written as $AB \equiv A_\mu B_\mu = \mathbf{AB} + A_4B_4 = \mathbf{AB} - A_0B_0$. A summation from 1 to 4 is implied over repeating Greek indices, and a summation from 1 to 3 – over repeating Latin indices. In this notation equation (2) can be written in the explicitly covariant form

$$\left(\frac{\partial^2}{\partial x_\mu^2} - \lambda\!\!\!^{-}{}^{-2}\right)\psi(x) = 0. \tag{3}$$

From this equation there follows the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{4}$$

where

$$\rho = i\frac{\lambda\!\!\!^{-}}{2c}\left(\psi^*\frac{\partial \psi}{\partial t} - \psi\frac{\partial \psi^*}{\partial t}\right), \qquad \mathbf{j} = -i\frac{\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right). \tag{5}$$

Here the quantity ρ is not positively definite, and therefore cannot be interpreted as a probability density. In addition, the transition to a higher order equation leads to the emergence of new solutions. In connection with this, the problem arises of constructing the quantum relativistic equation for the complex field, which would have the meaning of a probability amplitude.

3 Relativistic quantum equation for a probability amplitude

In spite of the shortcomings noted above, let us consider equation (1) as the equation for the probability amplitude $\psi(x)$. The advantage of this equation, which to a great extent compensates for the difficulties noted above, consists in the possibility of interpreting the complex field as a probability amplitude. In this equation the sign in front of the root can be chosen arbitrarily, since this will not affect the physical results, but only affect the form of the time dependence of the wave function for stationary states $\psi \sim \exp\big(-i\sigma Et/\hbar\big)$. In order for equation (1) to have the generally accepted form of the Schrödinger equation in the nonrelativistic limit, we will assume in (1) $\sigma = +1$. The equation

$$i\hbar\frac{\partial\psi}{\partial t} = \sqrt{m^2c^4 - \hbar^2c^2\Delta}\,\psi \tag{6}$$

does not have an explicitly covariant form, but leads to the continuity equation for the probability density $|\psi|^2$:

$$\frac{\partial|\psi|^2}{\partial t} + \nabla\cdot\mathbf{j} = 0, \tag{7}$$

where

$$\nabla\cdot\mathbf{j} = i\frac{mc^2}{\hbar}\Big(\psi^*\sqrt{1-\lambda^2\Delta}\,\psi - \psi\sqrt{1-\lambda^2\Delta}\,\psi^*\Big), \tag{8}$$

and $\mathbf{j}$ has the meaning of the probability flux density. The continuity equation (7) for the probability density can be written in 4-dimensional form

$$\frac{\partial j_\mu(x)}{\partial x_\mu} = 0, \tag{9}$$

where $j_\mu \equiv \big(\mathbf{j}, ic|\psi|^2\big)$ is a 4-vector of flux density, so that it has the same form in any inertial reference system. Thus, the equation for the probability amplitude (6), written in an arbitrary reference system, leads to the Lorentz covariant form of the probability conservation law (9). Since under the Lorentz transformation $x_\mu \to x'_\mu = a_{\mu\nu}x_\nu$, where $a_{\mu\nu}a_{\mu\nu'} = \delta_{\nu\nu'}$, the 4-vector is transformed according to the law $j'_\mu(x') = a_{\mu\nu}j_\nu(x)$, then the probability

density and the probability flux density in different systems are connected by the relations

$$ic|\psi'(x')|^2 = ic|\psi(x)|^2 + a_{4m}j_m(x),$$
$$j_k'(x') = a_{k4}\, ic|\psi(x)|^2 + a_{km}j_m(x). \tag{10}$$

Let us represent the complex field as the sum of the real and imaginary parts

$$\psi(x) = \frac{1}{\sqrt{2}}\left[\psi'(x) + i\psi''(x)\right]. \tag{11}$$

Then equation (6) turns out to be equivalent to a system of two equations for the real fields ψ' and ψ'':

$$\hbar\dot{\psi}' = \sqrt{m^2c^4 - \hbar^2c^2\Delta}\ \psi'', \qquad \hbar\dot{\psi}'' = -\sqrt{m^2c^4 - \hbar^2c^2\Delta}\ \psi'. \tag{12}$$

Here and henceforth we also use the notation $\dot{\psi} \equiv \partial\psi/\partial t$. One of the functions, for example ψ'', can be eliminated, and then we arrive at the explicitly covariant Klein-Gordon-Fock equation for the real part

$$\ddot{\psi}' - c^2\Delta\psi' + c^2\lambda\!\!\!^{-}{}^{-2}\psi' = 0. \tag{13}$$

Along with this, the evolution of the imaginary part is determined by the second equation (12). Similarly, the Klein-Gordon-Fock equation can be obtained for the imaginary part of the function (11).

Let us consider the meaning of the expression $\sqrt{1 - \lambda\!\!\!^{-}{}^2\Delta}\ \psi$, containing the square root of the Laplace operator. For $x^2 < 1$ the following expansions are valid [6]

$$\sqrt{1 \pm x} = 1 - \sum_{n=1}^{\infty}(\mp)^n a_n x^n,$$
$$a_n \equiv \frac{(2n-3)!!}{(2n)!!}, \quad a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{8}, \quad a_3 = \frac{3}{48}, \tag{14}$$

$$(1 \pm x)^{-1/2} = 1 + \sum_{n=1}^{\infty}(\mp)^n b_n x^n,$$
$$b_n \equiv \frac{(2n-1)!!}{(2n)!!}, \quad b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{8}, \quad b_3 = \frac{15}{48}, \tag{15}$$

where $(2n)!! = 2 \cdot 4 \cdot 6 \ldots 2n = 2^n n!$, $(2n+1)!! = 1 \cdot 3 \cdot 5 \ldots (2n+1) = \frac{2^{n+1}}{\sqrt{n}} \Gamma(n+3/2)$. With allowance for these representations, by the expression with the square root from the operator we will understand the following expansion

$$\sqrt{1 - \lambda\!\!\!^{-2} \Delta}\ \psi = \left(1 - \sum_{n=1}^{\infty} a_n \lambda\!\!\!^{-2n} \Delta^n\right) \psi. \tag{16}$$

This expansion is valid under the condition $|\lambda\!\!\!^{-2} \Delta \psi| < 1$, which is assumed to be satisfied in the following. This means that one considers the states in which the wave function changes over distances greater than the Compton length. The opposite ultrarelativistic case can be considered in a similar way. It should be noted, however, that it is of rather methodological interest, since in this limit the possibility of transformation of particles should be taken into account, which can only be done by passing to the representation of secondary quantization.

Taking (16) into account, we obtain a formula for the probability flux density. The divergence (8) can be written in the form

$$\nabla \cdot \mathbf{j} = -i \frac{mc^2}{\hbar} \sum_{n=1}^{\infty} a_n \lambda\!\!\!^{-2n} \left(\psi^* \Delta^n \psi - \psi \Delta^n \psi^*\right). \tag{17}$$

From here it follows that the probability flux density has the form

$$\mathbf{j} = \sum_{n=1}^{\infty} \mathbf{j}^{(n)},$$

$$\mathbf{j}^{(n)} = -ica_n \lambda\!\!\!^{-2n-1} \sum_{\alpha=1}^{n} \left[\left(\Delta^{\alpha-1} \psi^*\right)\left(\nabla \Delta^{n-\alpha} \psi\right) - \left(\Delta^{\alpha-1} \psi\right)\left(\nabla \Delta^{n-\alpha} \psi^*\right)\right]. \tag{18}$$

Everywhere it is assumed that the Laplacian to the zeroth power is equal to unity $\Delta^0 \equiv 1$.

4 Lagrangian formalism

Equation (6) can be obtained within the Lagrange formalism, if we choose the Lagrangian in the form

$$L = \frac{\psi^*}{2}\left(i\hbar\dot{\psi} - mc^2\sqrt{1-\lambda^2\Delta}\ \psi\right) - \frac{\psi}{2}\left(i\hbar\dot{\psi}^* + mc^2\sqrt{1-\lambda^2\Delta}\ \psi^*\right). \tag{19}$$

Lagrangian (19) is a function of variables ψ, ψ^*, $\dot{\psi}, \dot{\psi}^*$ and derivatives with respect to spatial coordinates $\Delta\psi, \Delta^2\psi, \ldots \Delta^n\psi, \ldots \Delta\psi^*, \Delta^2\psi^*, \ldots \Delta^n\psi^*, \ldots$. In this case, the Euler-Lagrange equations have the form

$$\frac{\partial L}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial L}{\partial\dot{\psi}} + \Delta\frac{\partial L}{\partial\Delta\psi} + \ldots + \Delta^n\frac{\partial L}{\partial\Delta^n\psi} + \ldots = 0, \tag{20}$$

and a similar equation with the replacement $\psi \to \psi^*$. A substitution of the Lagrangian (19) into (20) leads to Eq. (6).

In the absence of time-dependent external fields, from the requirement of invariance of the Lagrangian under the time shift $t \to t + t_0$ and $\psi(t) \to \psi'(t + t_0)$, there follows the continuity equation for the energy density

$$\frac{\partial \mathrm{H}}{\partial t} + \nabla \cdot \mathbf{j}_E = 0, \tag{21}$$

where the energy density

$$\mathrm{H} \equiv \dot{\psi}\frac{\partial L}{\partial\dot{\psi}} + \dot{\psi}^*\frac{\partial L}{\partial\dot{\psi}^*} - L = \frac{mc^2}{2}\left[\psi^*\sqrt{1-\lambda^2\Delta}\ \psi + \psi\sqrt{1-\lambda^2\Delta}\ \psi^*\right]. \tag{22}$$

The energy flux density is given by the formulas

$$\mathbf{j}_E = \sum_{n=1}^{\infty}\mathbf{j}_E^{(n)},$$

$$\begin{aligned}\mathbf{j}_E^{(n)} = \sum_{\alpha=1}^{n}\Big[&(\Delta^{n-\alpha}\dot{\psi})(\nabla\Delta^{\alpha-1}L_n) - (\nabla\Delta^{n-\alpha}\dot{\psi})(\Delta^{\alpha-1}L_n) + \\ &+ (\Delta^{n-\alpha}\dot{\psi}^*)(\nabla\Delta^{\alpha-1}L_n^*) - (\nabla\Delta^{n-\alpha}\dot{\psi}^*)(\Delta^{\alpha-1}L_n^*)\Big].\end{aligned} \tag{23}$$

Here we used the notation

$$L_n \equiv \frac{\partial L}{\partial \Delta^n \psi} = mc^2 a_n \lambda^{2n} \frac{\psi^*}{2}, \quad L_n^* \equiv \frac{\partial L}{\partial \Delta^n \psi^*} = mc^2 a_n \lambda^{2n} \frac{\psi}{2},$$
$$\Delta^\alpha L_n = mc^2 a_n \lambda^{2n} \frac{\Delta^\alpha \psi^*}{2}, \quad \Delta^\alpha L_n^* = mc^2 a_n \lambda^{2n} \frac{\Delta^\alpha \psi}{2}. \tag{24}$$

Taking (24) into account, we have the final expression for the energy flux density

$$\mathbf{j}_E^{(n)} = \frac{1}{2} mc^2 a_n \lambda^{2n} \sum_{\alpha=1}^{n} \Big[(\Delta^{n-\alpha} \dot{\psi})(\nabla \Delta^{\alpha-1} \psi^*) - (\nabla \Delta^{n-\alpha} \dot{\psi})(\Delta^{\alpha-1} \psi^*) + $$
$$+ (\Delta^{n-\alpha} \dot{\psi}^*)(\nabla \Delta^{\alpha-1} \psi) - (\nabla \Delta^{n-\alpha} \dot{\psi}^*)(\Delta^{\alpha-1} \psi) \Big]. \tag{25}$$

The condition of invariance of the Lagrangian with respect to spatial translations $\mathbf{x} \to \mathbf{x}' = \mathbf{x} + \mathbf{x}_0$ leads to the continuity equation for the momentum density:

$$\pi_i + \nabla_k \sigma_{ik} = 0, \tag{26}$$

where the momentum density π_i and the flux density of i-component of momentum σ_{ik} are given by the formulas

$$\pi_i = -\nabla_i \psi^* \frac{\partial L}{\partial \dot{\psi}^*} - \nabla_i \psi \frac{\partial L}{\partial \dot{\psi}} = -\frac{i\hbar}{2} (\psi^* \nabla_i \psi - \psi \nabla_i \psi^*), \tag{27}$$

$$\sigma_{ik} = \sum_{n=1}^{\infty} \sigma_{ik}^{(n)},$$

$$\sigma_{ik}^{(n)} = -\sum_{\alpha=1}^{n} \Big[(\Delta^{n-\alpha} \nabla_i \psi)(\nabla_k \Delta^{\alpha-1} L_n) - (\nabla_k \Delta^{n-\alpha} \nabla_i \psi)(\Delta^{\alpha-1} L_n) + \tag{28}$$
$$+ (\Delta^{n-\alpha} \nabla_i \psi^*)(\nabla_k \Delta^{\alpha-1} L_n^*) - (\nabla_k \Delta^{n-\alpha} \nabla_i \psi^*)(\Delta^{\alpha-1} L_n^*) \Big].$$

With allowance for (24), the last formula takes the form

$$\sigma_{ik}^{(n)} = -\frac{1}{2} mc^2 a_n \lambda^{2n} \times$$
$$\times \sum_{\alpha=1}^{n} \Big[(\Delta^{n-\alpha} \nabla_i \psi)(\nabla_k \Delta^{\alpha-1} \psi^*) - (\nabla_k \Delta^{n-\alpha} \nabla_i \psi)(\Delta^{\alpha-1} \psi^*) + \tag{29}$$
$$+ (\Delta^{n-\alpha} \nabla_i \psi^*)(\nabla_k \Delta^{\alpha-1} \psi) - (\nabla_k \Delta^{n-\alpha} \nabla_i \psi^*)(\Delta^{\alpha-1} \psi) \Big].$$

Lagrangian (19) is also invariant under the phase transformation

$$\psi(x) \to \tilde{\psi}(x) = \psi(x)e^{i\alpha}, \tag{30}$$

where the parameter α does not depend on coordinates and time. A consequence of this phase symmetry is the probability conservation law

$$\begin{aligned}&\left(\psi\frac{\partial L}{\partial\dot{\psi}} - \psi^*\frac{\partial L}{\partial\dot{\psi}^*}\right) + \\ &+\nabla_i\sum_{n=1}^{\infty}\sum_{\alpha=1}^{n}\Big[\big(\nabla_i\Delta^{n-\alpha}\psi\big)\Delta^{\alpha-1}L_n - \big(\Delta^{n-\alpha}\psi\big)\nabla_i\Delta^{\alpha-1}L_n - \\ &-\big(\nabla_i\Delta^{n-\alpha}\psi^*\big)\Delta^{\alpha-1}L_n^* + \big(\Delta^{n-\alpha}\psi^*\big)\nabla_i\Delta^{\alpha-1}L_n^*\Big] = 0.\end{aligned} \tag{31}$$

This equation, with taking into account formulas (24), coincides with the continuity equation (7) for the probability density which is obtained directly from equation (6).

5 Interaction with the electromagnetic field

The interaction with the electromagnetic field is enabled using the well-known derivative substitution

$$\frac{\partial}{\partial x_\mu} \to \frac{\partial}{\partial x_\mu} + i\frac{e}{\hbar c}A_\mu, \tag{32}$$

where $e = \pm|e|$, $A_\mu \equiv \big(\mathbf{A}, A_4 = i\Phi\big)$ is a 4-vector potential of the electromagnetic field. In three-dimensional notation, the substitution (32) is equivalent to the following substitutions:

$$\begin{aligned}&\nabla \to \nabla + \frac{e}{\hbar c}\mathbf{A}, \qquad i\hbar\frac{\partial}{\partial t} \to i\hbar\frac{\partial}{\partial t} + e\Phi, \\ &\Delta \to \Delta_A \equiv \Delta + \frac{e}{\hbar c}\nabla\mathbf{A} + \frac{2e}{\hbar c}\mathbf{A}\nabla + \left(\frac{e}{\hbar c}\right)^2\mathbf{A}^2.\end{aligned} \tag{33}$$

As a result, we obtain the quantum relativistic equation for the probability amplitude in the electromagnetic field:

$$i\hbar\frac{\partial\psi}{\partial t} - U(x)\,\psi = \sqrt{m^2c^4 - \hbar^2c^2\Delta_A}\;\psi, \tag{34}$$

where $U(x) = -e\Phi(x)$ is the potential energy in an external scalar field.

6 Schrödinger equation with the relativistic correction

Let us write the Schrödinger equation with the scalar potential, taking into account the main relativistic correction. In this approximation

$$\sqrt{1-\lambdabar^2\Delta}\;\psi \approx \left(1-\frac{1}{2}\,\lambdabar^2\Delta-\frac{1}{8}\,\lambdabar^4\Delta^2\right)\psi. \tag{35}$$

Then equation (34) will take the form

$$i\hbar\,\frac{\partial\psi}{\partial t} = U(x) + mc^2\left(1-\frac{1}{2}\,\lambdabar^2\Delta-\frac{1}{8}\,\lambdabar^4\Delta^2\right)\psi. \tag{36}$$

In order to proceed to the nonrelativistic limit, it is convenient to pass to the wave function $\chi(x)$, which differs from the original function by a time-dependent phase factor

$$\psi(x) = \chi(x)\,e^{-i\frac{mc^2}{\hbar}\,t}. \tag{37}$$

Then equation (36) will take the form

$$i\hbar\,\frac{\partial\chi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\chi + U(x)\,\chi - \frac{\hbar^4}{8m^3c^2}\Delta^2\chi. \tag{38}$$

Here the last term is the relativistic correction. The probability flux density (18) $\mathbf{j} = \mathbf{j}^{(1)}+\mathbf{j}^{(2)}$ is the sum of the nonrelativistic part $\mathbf{j}^{(1)}$ and the relativistic correction $\mathbf{j}^{(2)}$:

$$\begin{aligned} \mathbf{j}^{(1)} &= -\frac{i\hbar}{2m}(\chi^*\nabla\chi - \chi\nabla\chi^*), \\ \mathbf{j}^{(2)} &= -\frac{i\hbar}{8m}\lambdabar^2(\chi^*\nabla\Delta\chi - \chi\nabla\Delta\chi^* + \Delta\chi^*\nabla\chi - \Delta\chi\nabla\chi^*). \end{aligned} \tag{39}$$

The energy density (22) and the energy flux density $\mathbf{j}_E = \mathbf{j}_E^{(1)} + \mathbf{j}_E^{(2)}$ (25) in the considered approximation are given by the formulas

$$\mathrm{H} = mc^2|\chi|^2 - \frac{\hbar^2}{4m}(\chi^*\Delta\chi + \chi\Delta\chi^*) - \frac{\hbar^2}{16m}\lambdabar^2(\chi^*\Delta^2\chi + \chi\Delta^2\chi^*), \tag{40}$$

$$\mathbf{j}_E^{(1)} = \frac{\hbar^2}{4m}(\dot{\chi}^*\nabla\chi + \dot{\chi}\nabla\chi^* - \chi^*\nabla\dot{\chi} - \chi\nabla\dot{\chi}^{*'} + i\frac{\hbar}{2m}mc^2(\chi^*\nabla\chi - \chi\nabla\chi^*),$$

$$\mathbf{j}_E^{(2)} = \frac{\hbar^2}{16m}\lambda\!\!\!^{-2}\Big[(\nabla\chi^*)(\Delta\dot{\chi}) + (\Delta\dot{\chi}^*)(\nabla\chi) + (\nabla\Delta\chi^*)\dot{\chi} + \dot{\chi}^*(\nabla\Delta\chi) -$$
$$-\chi^*(\nabla\Delta\dot{\chi}) - (\nabla\Delta\dot{\chi}^*)\chi - (\Delta\chi^*)(\nabla\dot{\chi}) - (\nabla\dot{\chi}^*)(\Delta\chi)\Big] -$$
$$-i\frac{\hbar^2}{8m}mc^2\lambda\!\!\!^{-2}\Big[(\nabla\chi^*)(\Delta\chi) - (\Delta\chi^*)(\nabla\chi) + (\nabla\Delta\chi^*)\chi - \chi^*(\nabla\Delta\chi)\Big]. \tag{41}$$

As we see, in the continuity equation for energy, even in the nonrelativistic limit, it is necessary to take into account the rest energy of the particle.

The momentum density (27) is expressed through the probability flux density in the nonrelativistic approximation by the relation $\boldsymbol{\pi} = m\mathbf{j}^{(1)}$. The flux density of i-component of momentum σ_{ik} is given by the formulas

$$\sigma_{ik} = \sigma_{ik}^{(1)} + \sigma_{ik}^{(2)},$$

$$\sigma_{ik}^{(1)} = \frac{\hbar^2}{4m}\Big[\chi^*(\nabla_i\nabla_k\chi) + \chi(\nabla_i\nabla_k\chi^*) - \nabla_i\chi^*\nabla_k\chi - \nabla_i\chi\nabla_k\chi^*\Big],$$

$$\sigma_{ik}^{(2)} = \frac{\hbar^2}{16m}\lambda\!\!\!^{-2}\times$$
$$\times\Big[\chi^*(\nabla_i\nabla_k\Delta\chi) + \chi(\nabla_i\nabla_k\Delta\chi^*) + \Delta\chi^*(\nabla_i\nabla_k\Delta\chi) + \Delta\chi(\nabla_i\nabla_k\Delta\chi^*) -$$
$$-(\nabla_i\Delta\chi^*)\nabla_k\chi - (\nabla_k\Delta\chi^*)\nabla_i\chi - \nabla_i\chi^*(\nabla_k\Delta\chi) - \nabla_k\chi^*(\nabla_i\Delta\chi)\Big]. \tag{42}$$

In equation (38) the last term should be considered as a small perturbation.

7 Evolution of the wave function in an unlimited space

Let us consider the evolution of the wave function in an unlimited medium. Let us move on to the new wave function $\chi(x)$ (37), such that $\psi(x) = \chi(x)\exp(-it/\tau_C)$, where $\tau_C \equiv \lambda\!\!\!^{-}/c = \hbar/mc^2$. Note that for an electron $\tau_C \approx 1.3\cdot 10^{-21}$ s. In this case equation (6) will take the form

$$i\tau_C\dot{\chi} = \left(\sqrt{1-\lambda\!\!\!^{-2}\Delta} - 1\right)\chi. \tag{43}$$

The equations for the real and imaginary parts of the wave function $\chi = (\chi' + i\chi'')/\sqrt{2}$ can be written in the form

$$\tau_C \dot{\chi}' = \left(\sqrt{1-\lambda^2\Delta} - 1\right)\chi'', \quad \tau_C \dot{\chi}'' = -\left(\sqrt{1-\lambda^2\Delta} - 1\right)\chi'. \tag{44}$$

There holds the normalization condition

$$\frac{1}{2}\int d\mathbf{x}\left[\chi'^2(\mathbf{x},t) + \chi''^2(\mathbf{x},t)\right] = 1. \tag{45}$$

Let us consider the evolution of the wave function in an unlimited space, provided that at the initial moment $t = 0$ there are given the functions $\chi'(\mathbf{x},0)$ and $\chi''(\mathbf{x},0)$, normalized by the condition (45) . From equations (44), it follows a second order equation in time for the function χ':

$$\tau_C^2 \ddot{\chi}' + \left(\sqrt{1-\lambda^2\Delta} - 1\right)^2 \chi' = 0. \tag{46}$$

We expand the required functions into the Fourier integral

$$\chi'(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int \chi_k'(t)\, e^{i\mathbf{kx}} d\mathbf{k}, \quad \chi''(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int \chi_k''(t)\, e^{i\mathbf{kx}} d\mathbf{k}. \tag{47}$$

Substituting these expansions into (46) and the second equation (44), we obtain

$$\ddot{\chi}' + \omega_k^2 \chi_k' = 0, \qquad \dot{\chi}'' + \omega_k \chi_k' = 0, \tag{48}$$

where

$$\omega_k = \left(\sqrt{1+\lambda^2 k^2} - 1\right)\frac{1}{\tau_C}. \tag{49}$$

In the nonrelativistic limit $\lambda k \ll 1$

$$\omega_k \approx \frac{\lambda^2 k^2}{2\tau_C} = \frac{\hbar k^2}{2m}. \tag{50}$$

The group velocity, which determines the speed of propagation of a wave-particle

$$\nu_g = \frac{d\omega_k}{dk} = \lambda \frac{ck}{\sqrt{1+\lambda^2 k^2}}, \tag{51}$$

cannot exceed the speed of light. A calculation of the group velocity within the framework of ordinary nonrelativistic quantum mechanics by formula

(50) gives $\nu_g = \hbar k/m$, so that with an increase of k or a decrease of the de Broglie wavelength the speed of signal propagation can be arbitrarily large.

Solutions of equations (48) have the form

$$\chi'_k(t) = A_k e^{-i\omega_k t} + B_k e^{i\omega_k t}, \qquad \chi''_k(t) = -iA_k e^{-i\omega_k t} + iB_k e^{i\omega_k t}. \tag{52}$$

From the conditions for reality of the functions χ', χ'' it follows $\chi'_k = \chi'^*_{-k}$ and $\chi''_k = \chi''^*_{-k}$, so that $B_k = A^*_{-k}$. The normalization condition (45) gives

$$\int d\mathbf{k}\left(|A_k|^2 + |A_{-k}|^2\right) = 1. \tag{53}$$

The coefficients in (52) can be expressed through the initial conditions

$$A_k = \frac{1}{2}\left(\chi'_k(0) + i\chi''_k(0)\right), \quad A^*_{-k} = \frac{1}{2}\left(\chi'_k(0) - i\chi''_k(0)\right), \tag{54}$$

so that

$$\begin{aligned} \chi'_k(t) &= \chi'_k(0)\cos\omega_k t + \chi''_k(0)\sin\omega_k t, \\ \chi''_k(t) &= -\chi'_k(0)\sin\omega_k t + \chi''_k(0)\cos\omega_k t\,. \end{aligned} \tag{55}$$

Taking into account the relations

$$\chi'_k(0) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}\, e^{-i\mathbf{kx}}\chi'(\mathbf{x},0), \quad \chi''_k(0) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}\, e^{-i\mathbf{kx}}\chi''(\mathbf{x},0), \tag{56}$$

we find solutions which determine the real and imaginary parts of the wave function at an arbitrary moment of time from their values at the initial moment of time

$$\begin{aligned} \chi'(\mathbf{x},t) &= \int d\mathbf{x}'\big[J_C(\mathbf{x}-\mathbf{x}',t)\chi'(\mathbf{x}',0) + J_S(\mathbf{x}-\mathbf{x}',t)\chi''(\mathbf{x}',0)\big], \\ \chi''(\mathbf{x},t) &= \int d\mathbf{x}'\big[-J_S(\mathbf{x}-\mathbf{x}',t)\chi'(\mathbf{x}',0) + J_C(\mathbf{x}-\mathbf{x}',t)\chi''(\mathbf{x}',0)\big], \end{aligned} \tag{57}$$

where

$$\begin{aligned} J_C(\mathbf{x}-\mathbf{x}',t) &= \frac{1}{(2\pi)^3}\int d\mathbf{k}\cos\omega_k t\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}, \\ J_S(\mathbf{x}-\mathbf{x}',t) &= \frac{1}{(2\pi)^3}\int d\mathbf{k}\sin\omega_k t\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}. \end{aligned} \tag{58}$$

Using the expansion of a plane wave in terms of spherical functions and integrating over angles, we obtain

$$\begin{aligned} J_C(\mathbf{x}-\mathbf{x}',t) &\equiv J_C(|\mathbf{x}-\mathbf{x}'|,t) = \frac{1}{2\pi^2}\int_0^\infty dk\,k^2\cos(\omega_k t)\,j_0\big(k|\mathbf{x}-\mathbf{x}'|\big), \\ J_S(\mathbf{x}-\mathbf{x}',t) &\equiv J_S(|\mathbf{x}-\mathbf{x}'|,t) = \frac{1}{2\pi^2}\int_0^\infty dk\,k^2\sin(\omega_k t)\,j_0\big(k|\mathbf{x}-\mathbf{x}'|\big), \end{aligned} \tag{59}$$

where $j_0(x) = \sin x/x$ is the spherical Bessel function.

Let us express the sought functions at the initial moment $t = 0$ in terms of the modulus and phase

$$\chi'(\mathbf{x},0) = \sqrt{2}\,|\psi(\mathbf{x},0)|\cos\theta(\mathbf{x},0), \quad \chi''(\mathbf{x},0) = \sqrt{2}\,|\psi(\mathbf{x},0)|\sin\theta(\mathbf{x},0). \tag{60}$$

As the initial condition we choose the Gaussian distribution for the probability density

$$|\psi(r,0)|^2 = \frac{1}{\pi^{3/2}\sigma^3}\,e^{-r^2/\sigma^2}, \tag{61}$$

where $r \equiv |\mathbf{x}|$, and the parameter σ determines the width of the wave packet, so that as σ decreases the distribution (61), satisfying the normalization condition $4\pi\int_0^\infty |\psi(r,0)|^2 r^2 dr = 1$, approaches the delta function. For the initial phase in (60) we choose $\theta(\mathbf{x},0) = 0$, so that for the real and imaginary parts we have

$$\chi'(r,0) = \sqrt{2}\,|\psi(r,0)| = \frac{\sqrt{2}}{\pi^{3/4}\sigma^{3/2}}\,e^{-r^2/(2\sigma^2)}, \quad \chi''(r,0) = 0. \tag{62}$$

In the calculation it is helpful to use the well-known integral [6]

$$\int_0^\infty e^{-a^2t^2} t^{\nu+3/2} j_{\nu-1/2}(bt)\,dt = \sqrt{\frac{\pi}{2}}\frac{b^{\nu-1/2}}{(2a^2)^{\nu+1}}\,e^{-\frac{b^2}{4a^2}}. \tag{63}$$

As a result, we obtain a solution in the form

$$\begin{aligned} \chi'(r,t) &= \frac{\sqrt{2}}{\pi^{3/4}}\sqrt{\frac{2}{\pi}}\,\sigma^{3/2}\int_0^\infty dk\,k^2\cos(\omega_k t)\,j_0(kr)\,e^{-\frac{k^2\sigma^2}{2}}, \\ \chi''(r,t) &= -\frac{\sqrt{2}}{\pi^{3/4}}\sqrt{\frac{2}{\pi}}\,\sigma^{3/2}\int_0^\infty dk\,k^2\sin(\omega_k t)\,j_0(kr)\,e^{-\frac{k^2\sigma^2}{2}}. \end{aligned} \tag{64}$$

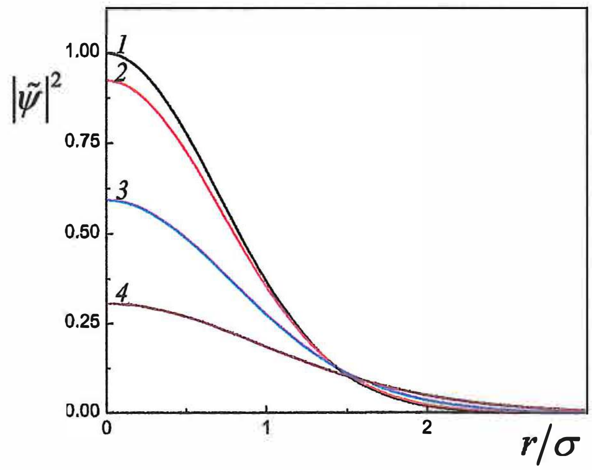

Figure 1: The spatial distribution of the probability density $|\tilde{\psi}|^2 \equiv \pi^{3/2}\sigma^3|\psi|^2$ for $\sigma/\lambda\!\!\!{}^{-} = 1$ at the moments of time $\tau = t/\tau_C$: *1* – 0; *2* – 0.5; *3* – 1.3; *4* – 2.0.

The spatial distribution of the probability density $|\tilde{\psi}|^2 \equiv \pi^{3/2}\sigma^3|\psi|^2 = \pi^{3/2}\sigma^3|\chi|^2$ at some moments of time is shown in Figure 1. As we see, a particle localized at the initial moment of time at the origin of coordinate system is gradually with equal probability spreading throughout space.

8 Relativistic correction in the oscillator theory

The Schrödinger equation for a one-dimensional along the x - axis quantum oscillator with account of the main relativistic correction (38) has the form

$$i\hbar\frac{\partial\chi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\chi}{\partial x^2} + \frac{m\omega^2x^2}{2}\chi - \frac{\hbar^4}{8m^3c^2}\frac{\partial^4\chi}{\partial x^4}. \tag{65}$$

In the stationary case, when $\chi(x,t) = \varphi(x)\exp\big(-iEt/\hbar\big)$, we have

$$E\varphi = -\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + \frac{m\omega^2x^2}{2}\varphi - \frac{\hbar^4}{8m^3c^2}\frac{d^4\varphi}{dx^4}. \tag{66}$$

It is convenient to introduce a dimensionless coordinate $\xi \equiv \left(\sqrt{m\omega/\hbar}\right)x$, and in result the equation takes the form

$$\varepsilon\varphi = -\varphi^{(II)} + \xi^2\varphi - \gamma\varphi^{(IV)}, \tag{67}$$

where differentiation occurs with respect to ξ, $\varepsilon \equiv 2E/\hbar\omega$, and

$$\gamma \equiv \frac{\hbar\omega}{4mc^2}. \tag{68}$$

In order to estimate this dimensionless parameter, we take $\hbar\omega \sim 1\,\mathrm{K} \approx 10^{-16}\,\mathrm{erg}$, $m \approx 10^{-23}\,\mathrm{g}$. As a result, we get a very small value $\gamma \sim 10^{-14}$.

The correction to n-th energy level is determined by the diagonal element of the perturbation operator taken over the unperturbed wave functions [5]

$$\varepsilon^{(1)} = -\gamma\langle n|\frac{d^4}{d\xi^4}|n\rangle. \tag{69}$$

The wave functions of an oscillator in the coordinate representation are expressed through Hermite polynomials [5]

$$\langle\xi|n\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\,e^{-\frac{\xi^2}{2}}H_n(\xi). \tag{70}$$

As a result, we find

$$\varepsilon^{(1)} = -\gamma\,\frac{3}{2}\,(n+1)\,(n+2). \tag{71}$$

Thus, the energy of the oscillator level with account of the main relativistic correction takes the form

$$E_n = \hbar\omega\left[n\left(1 - \frac{9}{4}\gamma\right) + \frac{1}{2}(1 - 3\gamma) - \frac{3}{4}\gamma n^2\right]. \tag{72}$$

As we see, the relativistic effect leads to a decrease of the ground and excited levels, as well as to a violation of the equidistance of the spectrum. The effect of violation of the equidistance property, leading to a decrease in the distance between higher levels, can be used for the experimental detection of relativistic effects in quantum mechanics.

9 Conclusion

The paper proposes the relativistic generalization of the Schrödinger equation for the complex function, which can be interpreted as a probability amplitude. Unlike the nonrelativistic equation, this equation allows propagation of signals only at a speed not exceeding the speed of light. The theory is also formulated within the Lagrangian formalism. The equations of conservation of the probability, energy and momentum of the complex field are obtained. The problem of evolution of a wave packet in an unlimited space is solved. The relativistic corrections to the energy levels of the harmonic oscillator are found. It is shown that relativistic effects lead to a shift and a violation of the level equidistance, which can be used for experimental detection of the relativistic effect in the probabilistic quantum theory.

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SINGLE TERM APPROXIMATIONS OF HYPERGEOMETRIC FUNCTIONS*

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Abstract

The hypergeometric functions occur naturally in variety of problems in analysis, statistics, operational research, theoretical and mathematical physics and engineering sciences. In the physical problems where these special functions occur, sometimes it is required to have their first approximations. In the advance analysis of a given problem, we need to dispose with some simple but sufficiently accurate algorithms for the approximations of the hypergeometric functions. In this paper we present some series and integral representations of them and discuss several simply-structured single term approximation formulae: for the Hubbell type radiation integrals, generalized hypergeometric functions and Appell's functions. We propose also a new approach to estimate the ${}_pF_q$-functions by relating them to three simple elementary functions. The notions generalized (multiple) fractional integrals and derivatives are used.

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1. Introduction

The *generalized hypergeometric functions* are defined by the series ([3],[17])

$$
\begin{aligned}
{}_pF_q(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; x) &= {}_pF_q \left[\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix} ; x \right] \\
&= \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k ... (a_p)_k}{(b_1)_k (b_2)_k ... (b_q)_k} \frac{x^k}{k!} , \quad \text{where } (a)_k = \Gamma(a+k)/\Gamma(a) .
\end{aligned} \tag{1}
$$

It is well known that the ${}_pF_q$-series converges for all finite x if $p \leq q$, for $|x| < 1$ if $p = q + 1$ and diverges for all $x \neq 0$ if $p > q + 1$.

According to [16], we consider three separate classes of ${}_pF_q$-functions:

• $p < q$. Every ${}_0F_1$-series is a Bessel function. Special cases include the trigonometric and hyperbolic sine and cosine functions. The generalized sine and cosine functions of order q and the so-called hyper-Bessel functions $J^{(q)}_{\nu_1,...,\nu_q}(x)$ ([1]) are also functions of this class, beeing: ${}_0F_{q-1}$. We call all such functions *generalized hypergeometric functions (g.h.f-s) of Bessel type.*

• $p = q$. The simplest examples are: ${}_0F_0$ – the exponential function and ${}_1F_1$ – the confluent hypergeometric series. Their special cases include the error functions, incomplete gamma functions, exponential and logarithmic integrals etc. Cases in which the series terminates include the Hermite, Laguerre and Bessel polynomials. All these special functions will be called *generalized hypergeometric functions (g.h.f-s) of confluent type.*

• $p = q + 1$. The ${}_1F_0$-function is the binomial series and ${}_2F_1$ is the well known Gauss hypergeometric function. Its special cases include logarithms, inverse trigonometric and hyperbolic functions, Legendre, Chebyshev, Gegenbauer and Jacobi polynomials and functions, incomplete beta functions, complete elliptic integrals of first and second kind, etc. Consider them as a class of the *Gauss type g.h.f-s.*

Exton [4] discussed a number of problems involving finite and infinite statistical distributions as well as a number of engineering problems which gave rise to integrals associated with hypergeometric functions of one and two variables. The enormous success of the theory of hypergeometric series

of one variable stimulated the development of a corresponding theory of two and more variables. Thus, we have the *Appell's series* which are defined in the following form [3]:

$$F_1(a,b,b';c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}\ , \quad \max\{|x|,|y|\} < 1; \tag{2}$$

$$F_2(a,b,b';c,c';x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m}{m!}\frac{y^n}{n!}\ , \quad |x|+|y| < 1; \tag{3}$$

$$F_3(a,a',b,b';c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \max\{|x|,|y|\} < 1; \tag{4}$$

$$F_4(a,b;c,c';x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n} \frac{x^m}{m!}\frac{y^n}{n!}\ , \quad \sqrt{|x|}+\sqrt{|y|} < 1, \tag{5}$$

where as usual, the denominator parameters c and c' are neither zeros nor negative integers.

For the Gauss hypergeometric function the so-called *Euler integral representation* is well known ([3],[17]):

$${}_2F_1(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\alpha-\beta)} \int_0^1 (1-u)^{\gamma-\beta-1}u^{\beta-1}(1-xu)^{-\alpha}du\ . \tag{6}$$

Appell gave some double integral representations of Euler type for the series F_1, F_2 and F_3, for example ([3]):

$$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y) = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}$$
$$\times \int_0^1\int_0^1 u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-ux-vy)^{-\alpha}du\ dv\ , \tag{7}$$

provided $\text{Re}(\gamma) > \text{Re}(\beta) > 0$, $\text{Re}(\gamma') > \text{Re}(\beta') > 0$.

Another representation of the same type is ([3]):

$$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y) = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\alpha)\Gamma(\gamma'-\alpha)} \int_0^1\int_0^1 v^{\beta-1}$$
$$\times (1-v)^{\gamma-\beta-1}(1-vx)^{\beta'-\alpha} w^{\alpha-1}(1-w)^{\gamma'-\alpha-1}(1-vx-wy)^{-\beta'}\, dv\, dw \tag{8}$$

with $\text{Re}(\gamma) > \text{Re}(\beta) > 0$, $\text{Re}(\gamma') > \text{Re}(\alpha) > 0$.

Several other integral representations have been found for the Appell's functions by various authors, all of them like (6),(7),(8) involving *fractional type integrals* (single or double, see Section 5 and [16],[19]). Picard pointed out that the series F_1 can be represented also by a single Eulerian type integral:

$$F_1(\alpha,\beta,\beta';\gamma;x,y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}$$
$$\times \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}\, du\ ,\ \text{Re}(\gamma) > \text{Re}(\alpha) > 0\ , \tag{9}$$

however there appears to be no single integral representation of this type for the Appell series F_4. Burchnall and Chaundy gave a double Eulerian integral for this function, similar to (7).

In all these representations it is assumed that $|x|$ and $|y|$ are small enough to make both series and integrals convergent.

There are many approximation formulas of different type, for the special functions of mathematical physics, and especially for generalized hypergeometric functions and integrals involving them. Polynomial, rational and Pade approximations for (1) are well known and often used (see e.g. Luke [17]). However, we would like to survey some single term approximations for hypergeometric functions and integrals, obtained recently by the authors and allowing fast and simple advanced evaluation of their values for practical use.

In Section 5 we propose also another approach to estimate the generalized hypergeometric functions by relating them to three simple elemen-

tary functions. The notions generalized (multiple) fractional integrals and derivatives ([16]) are used.

2. Single term approximations of radiation integrals

• **Hubbell rectangular source integral.** Hubbell et al. [9] have obtained a series expansion for the calculation of the radiation field generated by a plane isotropic rectangular source (plaque), in which the leading term is the integral

$$h(a,b) = \frac{\sigma}{4\pi} \int_0^b \operatorname{arctg}\left(\frac{a}{\sqrt{x^2+1}}\right) \frac{dx}{\sqrt{x^2+1}}, \quad a>0,\ b>0. \tag{10}$$

The above integral, called *Hubbell integral*, has found important applications in metrology situations where mechanical contact is not allowed. Typical examples can be found in a variety of medical, agricultural and industrial applications which make use of radiometric gauging and process control. In fact, these and other metrology devices are based on the principle of detecting radiation transmitted by a source of charged particles and photons. Integral (10) has been calculated, for selected values of the parameters, by means of the everywhere convergent series as given by Hubbell et al. [9] or by more computationally tractable, but empirical approximations, the latter devised in particular for engineering applications [8].

Kalla et al. [10] have defined and studied the more general radiation integral

$$H\begin{bmatrix} a,b,c,\lambda \\ \alpha,\beta,\gamma \end{bmatrix} = \frac{\sigma a}{4\pi} \int_0^b x^{\lambda}(x^2+c)^{-\alpha} {}_2F_1\left(\alpha,\beta;\gamma;-\frac{a^2}{x^2+c}\right) dx, \tag{11}$$

where $\gamma > \beta > 0$; $a,b,c > 0; \lambda > -1$ (if $b \to \infty$, then $-1 < \lambda < 1$) and ${}_2F_1(\alpha,\beta;\gamma;x)$ is the Gauss hypergeometric function. Observe that

$$H\begin{bmatrix} a,b,1,0 \\ 1,\frac{1}{2},\frac{3}{2} \end{bmatrix} = h(a,b)\ . \tag{12}$$

In Galue [5], Kalla, Galue, Kiryakova [12], Galue and Kiryakova [7] a generalization of the Hubbell integral and of (11) is studied in the following form:

$$
\begin{aligned}
I = I\begin{bmatrix} a, b, c, \lambda, \mu \\ \alpha, \beta, \gamma \end{bmatrix} \\
= \frac{\sigma a}{4\pi} \int_0^b x^\lambda (x^2+c)^{-\alpha} \left(1-\frac{x^2}{b^2}\right)^\mu {}_2F_1\left(\alpha, \beta; \gamma; -\frac{a^2}{x^2+c}\right) dx,
\end{aligned} \tag{13}
$$

with $c > 0$, $0 < a \leq b < \infty$, $\mu > -1$, $\lambda > -1$, and if $b \to \infty$, then $-1 < \lambda < 2\alpha - 2\mu - 1$. If $\mu = 0$, the function I reduces to $H\begin{bmatrix} a, b, c, \lambda \\ \alpha, \beta, \gamma \end{bmatrix}$ defined by equation (11).

In terms of the Appell's double hypergeometric function, this integral takes the form

$$
\begin{aligned}
I\begin{bmatrix} a, b, c, \lambda, \mu \\ \alpha, \beta, \gamma \end{bmatrix} = \frac{\sigma a}{4\pi} \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^b \int_0^1 x^\lambda (1-t)^{\gamma-\beta-1} \left(1-\frac{x^2}{b^2}\right) \\
\times t^{\beta-1}(x^2+c+a^2t)^{-\alpha} dt\, dx\ .
\end{aligned} \tag{14}
$$

From (14) and using the approximation formula

$$
(1+x/r)^{\alpha-1} \simeq (1-x)^{-(\alpha-1)/r} \tag{15}
$$

which in the neighbourhood of $x = 0$ is a better approximation to $(1 + x/r)^{\alpha-1}$ than just the unity ([13]), in [6] we have obtained a *single term approximation for* I:

$$
\begin{aligned}
I = I\begin{bmatrix} a, b, c, \lambda, \mu \\ \alpha, \beta, \gamma \end{bmatrix} \simeq \frac{\sigma a}{8\pi} \frac{b^{\lambda+1}}{c^\alpha} \frac{\Gamma(\frac{\lambda+1}{2})\Gamma(\mu+\alpha\frac{b^2}{c}+1)}{\Gamma(\frac{\lambda+3}{2}+\mu+\alpha\frac{b^2}{c})} \\
\times \frac{\Gamma(\gamma-\beta+\alpha\frac{a^2}{2c})\Gamma(\gamma)}{\Gamma(\gamma+\alpha\frac{a^2}{2c})\Gamma(\gamma-\beta)}, \quad \text{provided } a^2b^2/(2c) << 1\ .
\end{aligned} \tag{16}
$$

In view of the symmetry of the hypergeometric function, equation (16) can be written alternatively as

$$I = I\begin{bmatrix} a, b, c, \lambda, \mu \\ \alpha, \beta, \gamma \end{bmatrix} \simeq \frac{\sigma a}{8\pi}\, \frac{b^{\lambda+1}}{c^{\alpha}}\, \frac{\Gamma(\frac{\lambda+1}{2})\Gamma(\mu + \alpha\frac{b^2}{c} + 1)}{\Gamma(\frac{\lambda+3}{2} + \mu + \alpha\frac{b^2}{c})} \times \frac{\Gamma(\gamma - \alpha + \beta\frac{a^2}{2c})\Gamma(\gamma)}{\Gamma(\gamma + \beta\frac{a^2}{2c})\Gamma(\gamma - \alpha)} . \tag{17}$$

If in equation (17) $\mu = 0$, we obtain a *single term approximation* for the function $H\begin{bmatrix} a, b, c, \lambda \\ \alpha, \beta, \gamma \end{bmatrix}$, namely

$$H = H\begin{bmatrix} a, b, c, \lambda \\ \alpha, \beta, \gamma \end{bmatrix} \simeq \frac{\sigma a}{8\pi}\, \frac{b^{\lambda+1}}{c^{\alpha}}\, \frac{\Gamma(\frac{\lambda+1}{2})\Gamma(\alpha\frac{b^2}{c} + 1)}{\Gamma(\frac{\lambda+3}{2} + \alpha\frac{b^2}{c})} \times \frac{\Gamma(\gamma - \alpha + \beta\frac{a^2}{2c})\Gamma(\gamma)}{\Gamma(\gamma + \beta\frac{a^2}{2c})\Gamma(\gamma - \alpha)} \quad \text{with} \quad a^2b^2/(2c) << 1.$$

Then, the corresponding formula for the Hubbell integral is:

$$h(a,b) \simeq \frac{\sigma a}{2\pi}\, \frac{b}{2 + a^2}\, \frac{\Gamma(\frac{3}{2})\Gamma(b^2 + 1)}{\Gamma(\frac{3}{2} + b^2)} \quad \text{with} \quad a^2b^2/2 << 1.$$

Following a similar procedure, we can obtain an approximation formula for a more general radiation integral.

• **Single term approximation for the** $S_m^{(p,p)}\begin{bmatrix} a, b, c, \lambda \\ \alpha, (\alpha_p), (\beta_q) \end{bmatrix}$**-function.**

Saigo and Srivastava [18] have studied a family of integrals of the form

$$S = S_m^{(p,q)}\begin{bmatrix} a, b, c, \lambda \\ \alpha, (\alpha_p), (\beta_q) \end{bmatrix} = \frac{\sigma a}{4\pi} \int_0^b x^{\lambda}(x^m + c)^{-\alpha}\ {}_{p+1}F_q\begin{bmatrix} \alpha, & \alpha_1, \dots, \alpha_p; & -\dfrac{a^m}{x^m + c} \\ & \beta_1, \dots, \beta_q; & \end{bmatrix} dx\ , \tag{18}$$

where $\min\{a,b,c\}>0, \lambda\in(-1,1)$ and ${}_pF_q$ is defined by (1).

One immediate special case of (18) obtained when $p=q=1$ and $m=2$, is

$$S_2^{(1,1)}\begin{bmatrix} a,b,c,\lambda \\ \alpha,\beta,\gamma \end{bmatrix} = H\begin{bmatrix} a,b,c,\lambda \\ \alpha,\beta,\gamma \end{bmatrix} . \tag{19}$$

In [6] we have derived a *single term approximation for* $S_m^{(p,p)}(p=q)$ in the following form:

$$\begin{aligned} S = S_m^{(p,p)}\begin{bmatrix} a,b,c,\lambda \\ \alpha,(\alpha_p),(\beta_p) \end{bmatrix} &\simeq \frac{\sigma a}{4\pi}\frac{b^{\lambda+1}}{mc^{\alpha-\alpha_p}}\,(c+a^m)^{-\alpha_p} \\ &\times \frac{\Gamma\left(\frac{\lambda+1}{m}\right)\Gamma\left(\alpha\frac{b^m}{c}+1\right)}{\Gamma\left(\frac{\lambda+1}{m}+\alpha\frac{b^m}{c}+1\right)}\prod_{i=0}^{p-1}\frac{\Gamma\left(\alpha_i-\alpha_p\left(\frac{a^m}{c+a^m}\right)\right)\Gamma(\beta_{i+1})}{\Gamma\left(\beta_{i+1}-\alpha_p\left(\frac{a^m}{c+a^m}\right)\right)\Gamma(\alpha_i)} \end{aligned} \tag{20}$$

for $0<a^m/c<<1$ (or $a^m/c>>1$) and $(ab)^m/c^2<<1$.

Letting $p=1$ and $m=2$ in (20), we obtain for $\alpha_0=\alpha,\ \alpha_1=\beta,\ \beta_1=\gamma$, *another single term approximation* for the function $H\begin{bmatrix} a,b,c,\lambda \\ \alpha,\beta,\gamma \end{bmatrix}$ in the form:

$$\begin{aligned} H = H\begin{bmatrix} a,b,c,\lambda \\ \alpha,\beta,\gamma \end{bmatrix} &\simeq \frac{\sigma a}{8\pi}\frac{b^{\lambda+1}}{c^{\alpha-\beta}}\,(c+a^2)^{-\beta} \\ &\times \frac{\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(\alpha\frac{b^2}{c}+1\right)\ \Gamma\left(\alpha-\beta\left(\frac{a^2}{c+a^2}\right)\right)\Gamma(\gamma)}{\Gamma\left(\frac{\lambda+3}{2}+\alpha\frac{b^2}{c}\right)\ \ \Gamma\left(\gamma-\beta\left(\frac{a^2}{c+a^2}\right)\right)\Gamma(\alpha)}\ , \end{aligned} \tag{21}$$

for $0<a^2/c<<1$ (or $a^2/c>>1$) and $(ab)^2/c^2<<1$, and as a corollary, the *corresponding formula for the Hubbell integral*:

$$h(a,b)\simeq\frac{\sigma a}{8\pi}\frac{b}{(1+a^2)^{\frac{1}{2}}}\ \frac{\Gamma(\frac{1}{2})\Gamma(b^2+1)}{\Gamma(\frac{3}{2}+b^2)}\ \frac{\Gamma\left(1-\frac{1}{2}\left(\frac{a^2}{1+a^2}\right)\right)\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}-\frac{1}{2}\left(\frac{a^2}{1+a^2}\right))} \tag{22}$$

for $0<a^2<<1$ (or $a^2>>1$) and $(ab)^2<<1$.

The above single term approximations for the functions S and I are sufficiently accurate to allow a semi-quantitative assessment of the dependence of these functions on the parameters, which is usually an important first step in the physical applications.

3. Single term approximations for the Gauss type generalized hypergeometric functions ${}_{p+1}F_p$

Using successively the result [3]

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}{}_pF_q\left((a_p);(b_q);wx\right)dx = B(\alpha,\beta)\ {}_{p+1}F_{q+1}\left((a_p),\alpha;(b_q),\alpha+\beta;w\right) \tag{23}$$

$(a, \operatorname{Re}\alpha, \operatorname{Re}\beta > 0; p \le q+1)$, Kalla and Galue [11], Kiryakova [14] obtained for ${}_pF_q$, $p = q+1$ the following multiple integral representation:

$$\begin{aligned}
{}_{p+1}F_p\left[\begin{matrix}\alpha, & \alpha_1,\dots,\alpha_p; \\ & \beta_1,\dots,\beta_p;\end{matrix}\ x\right] &= \frac{1}{B(\alpha,\beta_1-\alpha)}\frac{1}{B(\alpha_1,\beta_2-\alpha_1)}\frac{1}{B(\alpha_2,\beta_3-\alpha_2)} \\
&\times\cdots\frac{1}{B(\alpha_{p-1},\beta_p-\alpha_{p-1})}\int_0^1\int_0^1\int_0^1 \underset{(p-4)}{\cdots}\int_0^1 (1-w)^{\alpha-1}w^{\beta_1-\alpha-1} \\
&\times(1-w_1)^{\alpha_1-1}w_1^{\beta_2-\alpha_1-1}(1-w_2)^{\alpha_2-1}w_2^{\beta_3-\alpha_2-1}\cdots(1-w_{p-1})^{\alpha_{p-1}-1} \\
&\times w_{p-1}^{\beta_p-\alpha_{p-1}-1}\left(1-x(1-w)(1-w_1)(1-w_2)\cdots(1-w_{p-1})\right)^{-\alpha_p} \\
&\times dw_{p-1}\cdots dw_2\,dw_1\,dw\ .
\end{aligned} \tag{24}$$

The approximation formula

$$\begin{aligned}
1-x(1-w)(1-w_1)(1-w_2)\cdots(1-w_{p-1}) &\simeq (1-x)(1+\frac{x}{1-x}w) \\
\times(1+\frac{x}{1-x}w_1)(1+\frac{x}{1-x}w_2)\cdots(1+\frac{x}{1-x}w_{p-1}) &\quad \text{for } 0<|x|<<1,
\end{aligned} \tag{25}$$

and (15) turn representation (24) into

$$
{}_{p+1}F_p\left[\begin{matrix}\alpha, & \alpha_1,...,\alpha_p; \\ & \beta_1,...,\beta_p;\end{matrix}\, x\right] \simeq \frac{1}{B(\alpha,\beta_1-\alpha)}\,\frac{1}{B(\alpha_1,\beta_2-\alpha_1)}\,\frac{1}{B(\alpha_2,\beta_3-\alpha_2)}
$$

$$
\times\cdots\frac{1}{B(\alpha_{p-1},\beta_p-\alpha_{p-1})}\,(1-x)^{-\alpha_p}\int_0^1\int_0^1\int_0^1\underset{(p-4)}{\cdots}\int_0^1(1-w)^{\alpha+\alpha_p(\frac{x}{1-x})-1}
$$

$$
\times\, w^{\beta_1-\alpha-1}(1-w_1)^{\alpha_1+\alpha_p(\frac{x}{1-x})-1}w_1^{\beta_2-\alpha_1-1}(1-w_2)^{\alpha_2+\alpha_p(\frac{x}{1-x})-1}w_2^{\beta_3-\alpha_2-1}
$$

$$
\times\cdots(1-w_{p-1})^{\alpha_{p-1}+\alpha_p(\frac{x}{1-x})-1}w_{p-1}^{\beta_p-\alpha_{p-1}-1}\;dw_{p-1}\cdots dw_2\,dw_1\,dw\;. \tag{26}
$$

Then, from the known rational approximation of ${}_pF_q$ (see Luke [17], §5.12, (1)-(4),(8)), the following *single term approximation* can be obtained:

$$
{}_{p+1}F_p\left[\begin{matrix}\alpha, & \alpha_1,...,\alpha_p; \\ & \beta_1,...,\beta_p;\end{matrix}\, x\right] \simeq (1-x)^{-\alpha_p}\prod_{i=0}^{p-1}\frac{\Gamma\left(\alpha_i+\alpha_p(\frac{x}{1-x})\right)\Gamma(\beta_{i+1})}{\Gamma\left(\beta_{i+1}+\alpha_p(\frac{x}{1-x})\right)\Gamma(\alpha_i)} \tag{27}
$$

with $0<|x|<<1, \mathrm{Re}\left(\alpha_i+\alpha_p(\frac{x}{1-x})\right)>0, \mathrm{Re}(\beta_{i+1})>0, i=0,1,2,...,p-1$.

Particular cases:

a) For $p=1$ we get for the Gauss hypergeometric function:

$$
{}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \gamma;\end{matrix}\, x\right] \simeq (1-x)^{-\beta}\,\frac{\Gamma\left(\alpha+\beta(\frac{x}{1-x})\right)\Gamma(\gamma)}{\Gamma\left(\gamma+\beta\left(\frac{x}{1-x}\right)\right)\Gamma(\alpha)} \tag{28}
$$

for $0<|x|<<1, \mathrm{Re}\left(\alpha+\beta\left(\frac{x}{1-x}\right)\right)>0, \mathrm{Re}(\gamma)>0$.

b) If $p=2$,

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\alpha,\alpha_1,\alpha_2; \\ \beta_1,\beta_2;\end{matrix}\, x\right] &\simeq (1-x)^{-\alpha_2} \\
&\times\frac{\Gamma\left(\alpha+\alpha_2\left(\frac{x}{1-x}\right)\right)\Gamma(\beta_1)}{\Gamma\left(\beta_1+\alpha_2\left(\frac{x}{1-x}\right)\right)\Gamma(\alpha)}\,\frac{\Gamma\left(\alpha_1+\alpha_2\left(\frac{x}{1-x}\right)\right)\Gamma(\beta_2)}{\Gamma\left(\beta_2+\alpha_2\left(\frac{x}{1-x}\right)\right)\Gamma(\alpha_1)}
\end{aligned} \tag{29}
$$

for $0<|x|<<1, \mathrm{Re}\left(\alpha+\alpha_2\left(\frac{x}{1-x}\right)\right)>0, \mathrm{Re}(\beta_1)>0, \mathrm{Re}\left(\alpha_1+\alpha_2\left(\frac{x}{1-x}\right)\right)$ $>0,\ \mathrm{Re}(\beta_2)>0.$

4. Single term approximations for the Appell's functions

For the Appell's hypergeometric functions of two variables Kalla and Galue [11] obtained the following *single term approximations*. From (9),

$$F_1(\alpha,\beta,\beta';\gamma;x,y)\simeq(1-x)^{-\beta}(1-y)^{-\beta'}\,\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\,\frac{\Gamma\left(\alpha+\frac{\beta x}{1-x}+\frac{\beta' y}{1-y}\right)}{\Gamma\left(\gamma+\frac{\beta x}{1-x}+\frac{\beta' y}{1-y}\right)} \tag{30}$$

with $\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0, \mathrm{Re}\left(\alpha+\frac{\beta x}{1-x}+\frac{\beta' y}{1-y}\right)>0$. From (7) and (8), assuming

$$(1-ax-by)\simeq(1-ax)(1-by)\quad \text{if}\ \ 0<|xy|<<1, \tag{31}$$

and using (15), we find similarly,

$$\begin{aligned}F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y)&\simeq(1-x)^{-\alpha}(1-y)^{-\alpha}\\&\times\frac{\Gamma(\gamma)}{\Gamma(\beta)}\,\frac{\Gamma(\gamma')}{\Gamma(\beta')}\,\frac{\Gamma\left(\beta+\frac{\alpha x}{1-x}\right)}{\Gamma\left(\gamma+\frac{\alpha x}{1-x}\right)}\,\frac{\Gamma\left(\beta'+\frac{\alpha y}{1-y}\right)}{\Gamma\left(\gamma'+\frac{\alpha y}{1-y}\right)},\end{aligned} \tag{32}$$

with $\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0, \mathrm{Re}(\gamma')>\mathrm{Re}(\beta')>0, 0<|xy|<<1$ and small α; respectively,

$$\begin{aligned}F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y)&\simeq(1-x)^{-\alpha}(1-y)^{-\alpha}\\&\times\frac{\Gamma(\gamma)}{\Gamma(\beta)}\,\frac{\Gamma(\gamma')}{\Gamma(\alpha)}\,\frac{\Gamma\left(\beta+\frac{\alpha x}{1-x}\right)}{\Gamma\left(\gamma+\frac{\alpha x}{1-x}\right)}\,\frac{\Gamma\left(\alpha+\frac{\beta' y}{1-y}\right)}{\Gamma\left(\gamma'+\frac{\beta' y}{1-y}\right)},\end{aligned} \tag{33}$$

with $\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma') > \mathrm{Re}(\alpha) > 0, 0 < |xy| << 1$, small β'.

Analogous formulas hold for F_3 and F_4.

Other single term approximations can be established using (15) and (31) again, for example:

$$F_1(\alpha, \beta, \beta'; \gamma; x, y) \simeq \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} \frac{\Gamma(\gamma-\alpha-\beta x-\beta' y)}{\Gamma(\gamma-\beta x-\beta' y)} \tag{34}$$

provided. $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0, x \simeq 0, y \simeq 0$, and

$$F_2(\alpha, \beta, \beta'; \gamma, \gamma'; x, y) \simeq \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\gamma-\beta)\Gamma(\gamma')-\Gamma(\beta')} \frac{\Gamma(\gamma-\beta-\alpha x)}{\Gamma(\gamma-\alpha x)} \frac{\Gamma(\gamma'-\beta'-\alpha y)}{\Gamma(\gamma'-\alpha y)} \tag{35}$$

provided $\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma') > \mathrm{Re}(\beta') > 0, 0 < |xy| << 1$. For the technical details see Galue, Kalla, Leubner [6].

5. Approximations of generalized hypergeometric functions via generalized fractional calculus

Formula (23) ([3]) turns to be also a corner-stone of the *new and unified approach to the generalized hypergeometric functions* (1), Kiryakova [15],[16]. All the ${}_pF_q$-functions are proved to be "generalized fractional integrals or derivatives" of *three basic elementary functions*, depending on whether $p < q, p = q$ or $p = q + 1$, namely:

$$\cos_{q-p+1}(x) \ , \quad x^{\alpha} \exp x \ , \quad x^{\alpha}(1-x)^{\beta}. \tag{36}$$

These new differintegral expressions can be used successfully *to reduce the problem of the ${}_pF_q$'s asymptotic behaviour to that of the well-known functions (36).*

The *generalized fractional calculus* developed by Kiryakova [16] is based on compositions of the so-called *Erdélyi-Kober operators of fractional integration (E.-K. fractional integrals)* (see [19]):

$$I_{\beta}^{\gamma,\delta} f(x) = \int_0^1 \frac{(1-\sigma)^{\delta-1}\sigma^{\gamma}}{\Gamma(\delta)} f(x\sigma^{\frac{1}{\beta}})\, d\sigma \ , \delta > 0, \gamma \in \boldsymbol{R} \ , \tag{37}$$

generalizing the Riemann-Liouville fractional integrals of order $\delta > 0$:

$$R^\delta f(x) = \int_0^x \frac{(x-t)^{\delta-1}}{\Gamma(\delta)} f(t)dt = x^\delta \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} f(x\sigma^{\frac{1}{\beta}})d\sigma = x^\delta I_1^{0,\delta} f(x).$$

Let $m \geq 1$ be integer, $\beta > 0$, $\gamma_1, ..., \gamma_m$ and $\delta_1 \geq 0, ..., \delta_m \geq 0$ be arbitrary real numbers. We define *multiple (m-tuple) Erdélyi-Kober (E.-K.) operators of integration of multiorder* $\delta = (\delta_1, ..., \delta_m)$ by means of the integral operators

$$I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) = \int_0^1 G_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (\gamma_k + \delta_k)_1^m \\ (\delta_k)_1^m \end{matrix} \right. \right] f(x\sigma^{\frac{1}{\beta}})\, d\sigma, \tag{38}$$

where the kernel-function $G_{m,m}^{m,0}$ is a special case $(m = p = q, n = 0)$ of the so-called *Meijer's G-function* (see [3],[16],[17]):

$$G_{p,q}^{m,n} \left[\sigma \left| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod\limits_{k=1}^{m} \Gamma(b_k - s) \prod\limits_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod\limits_{k=m+1}^{q} \Gamma(1 - b_k + s) \prod\limits_{j=n+1}^{p} \Gamma(a_j - s)} \sigma^s ds. \tag{39}$$

Then, each operator of the form

$$Rf(x) = x^{\beta\delta_0} I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) \quad \text{with arbitrary} \quad \delta_0 \geq 0, \tag{40}$$

is said to be a *generalized (m-tuple) fractional integral.*

It turns out that single integral operators (38) are equivalent to the multiple integrals (like these appearing in Section 3, (24))

$$I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) = I_\beta^{\gamma_m,\delta_m} \left\{ I_\beta^{\gamma_{m-1},\delta_{m-1}} \ldots \left(I_\beta^{\gamma_1,\delta_1} f(x) \right) \right\} = \left[\prod_{k=1}^m I_\beta^{\gamma_k,\delta_k} \right] f(x)$$

$$= \underbrace{\int_0^1 \cdots \int_0^1}_{m} \left[\prod_{k=1}^m \frac{(1-\sigma_k)^{\delta_k - 1} \sigma_k^{\gamma_k}}{\Gamma(\delta_k)} \right] f\left(x\sigma_1^{\frac{1}{\beta}} \ldots \sigma_m^{\frac{1}{\beta}} \right) d\sigma_1 \ldots d\sigma_m, \tag{41}$$

representing *compositions of finite number* $m > 1$ *of commuting E.-K. operators* $I_\beta^{\gamma_k,\delta_k}$, $k = 1, ..., m$ (Th. 1.2.10, [16]). This fact explains the name given above to (38). The Meijer's G-functions are generalized hypergeometric functions of very general nature; they include as special cases almost all the elementary and the special functions of mathematical physics. That is why, dealing with the single integral operators (38) instead of compositions (41), has a number of advantages. The use of the simple properties of special functions (39) (see [3],[17],[16, App.]) allows to develop a full chain of operational rules and theory of the generalized fractional calculus [16], based on the generalized fractional integrals (38),(39). The *generalized fractional derivatives*, corresponding to (38) are also introduced by means of explicit differintegral expressions, namely ([16], §1.5):

$$D_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) = D_\eta \, I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(x), \tag{42}$$

where $\eta_k = \delta_k$, if δ_k is integer, or $\eta_k = [\delta_k] + 1$ if δ_k is noninteger, $k = 1, ..., m$ and

$$D_\eta = \left[\prod_{r=1}^{m} \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta} \frac{x}{dx} + \gamma_r + j \right) \right].$$

Then, in suitable functional spaces like the space of analytic functions, $D_\beta^{(\gamma_k),(\delta_k)} I_\beta^{(\gamma_k),\delta_k} f(x) = f(x)$.

Both generalized fractional integrals (38),(39) and corresponding derivatives (42) will be called by the *commmon name "generalized fractional differintegrals"*. The following general proposition is derived in [16, Chapter 4]:

PROPOSITION 5.0. *All the generalized hypergeometric functions* ${}_pF_q$ *can be considered as generalized (q-tuple) fractional differintegrals (38), (42) of one of the elementary functions (36), depending on whether* $p < q, p = q, p = q + 1$.

To establish above proposition we use formula (23) stated in the following form.

LEMMA . *Let $|x| < \infty$ ($|x| < 1$ for $p = q+1$), then each ${}_pF_q$-function is an Erdélyi-Kober fractional integral or derivative of a ${}_{p-1}F_{q-1}$-function, namely:*

$$[\Gamma(a_p)/\Gamma(b_q)]\ {}_pF_q(a_1,...,a_p;b_1,...,b_q;x)$$
$$= \begin{cases} I_1^{a_p-1,b_q-a_p}\{{}_{p-1}F_{q-1}(a_1,...,a_{p-1};b_1,...,b_{q-1};x)\} & \text{if } b_q > a_p, \\ D_1^{b_q-1,a_p-b_q}\{{}_{p-1}F_{q-1}(a_1,...,a_{p-1};b_1,...,b_{q-1};x)\} & \text{if } b_q < a_p. \end{cases} \tag{43}$$

The three cases $p < q, p = q, p = q+1$ are to be considered separately, first of them being more complicated and involving some auxiliary definitions and results.

1st case: $p < q$

In [1] Delerue introduced a generalization of the Bessel function $J_\nu(x)$ for a multiindex $\nu = (\nu_1,...,\nu_m)$, $m \geq 1$:

$$J^{(m)}_{\nu_1,...,\nu_m}(x) = \frac{(x/m+1)^{\nu_1+...+\nu_m}}{\Gamma(\nu_1+1)...\Gamma(\nu_m+1)}\ {}_0F_m\left((\nu_k+1)_1^m; -(x/m+1)^{m+1}\right), \tag{44}$$

called a *hyper-Bessel function of order m*. As a special case, the so-called *(generalized) cosine function of order* $(m+1)$ follows:

$$\cos_{m+1}(x) = {}_0F_m\left((\frac{k}{m+1})_1^m; -(x/m+1)^{m+1}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k(m+1)}}{(k(m+1))!}, \tag{45}$$

generalizing the elementary cosine function $\cos x = \cos_2(x)$, $m = 1$.

In [2] Dimovski and Kiryakova proved a *generalization of the Poisson integral representation of the Bessel function*

$$J_\nu(x) = \frac{2}{\sqrt{\pi}}\ \frac{(\frac{x}{2})^\nu}{\Gamma(\nu+\frac{1}{2})}\ \int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\ \cos xt\ dt, \quad \nu > -\frac{1}{2}, \tag{46}$$

based on the Poisson-Dimovski transformation (see [16], Ch.3). In terms of the generalized fractional calculus, G- and ${}_pF_q$-functions, the generalized

Poisson integral ([16], Th. 4.1.1, Cor. 4.1.4) can be written in the modified form:

$$
\begin{aligned}
&{}_0F_m\left((b_k)_1^m; -x\right) \\
&= c\, I_{1,m}^{(k/(m+1)-1),(b_k-k/(m+1))}\{\cos_{m+1}\left((m+1)x^{1/(m+1)}\right)\} \\
&= c\int_0^1 G_{m,m}^{m,0}\left[\sigma \middle| \begin{matrix}(b_k)_1^m \\ \left(\frac{k}{m+1}\right)_1^m\end{matrix}\right] \sigma^{-1}\cos_{m+1}\left((m+1)(x\sigma)^{1/(m+1)}\right) d\sigma,
\end{aligned}
\tag{47}
$$

where $c = \sqrt{(m+1)/(2\pi)^m}\ \prod_{j=1}^m \Gamma(b_j)$ and the condition $b_k \geq k/(m+1), k = 1, ..., m$ is supposed. Otherwise, if some of the b_k's does not satisfy it, its corresponding component in the generalized fractional differintegral should be considered as an E.-K. derivative ([16], Th.4.1.6). As an illustration, if $b_k := \nu_k + 1 = k/(m+1) - \eta_k$ with integers $\eta_k > 0, k = 1, ..., m$, then (47) turns into a *purely differential expression for the "spherical" hyper-Bessel functions (44)* ([16],(4.1.42)), reducible in the case $m = 1$ to the spherical Bessel functions:

$$
J_{-\eta-\frac{1}{2}}(x) = \frac{(2x)^{\eta+\frac{1}{2}}}{\sqrt{\pi}}\ \frac{d\eta}{(dx^2)^\eta}\left\{\frac{\cos x}{x}\right\},\quad \eta = 0, 1, 2... \tag{48}
$$

Let us observe now that by p steps (23), i.e.(43), a ${}_pF_q$-function, $p < q$ can be reduced to a hyper-Bessel function, i.e. to a ${}_0F_{q-p}$-function:

$$
\begin{aligned}
&{}_pF_q\left((a_k)_1^p; (b_l)_1^q; x\right) \\
&= [\prod_{j=1}^p \frac{\Gamma(b_{q-p+j})}{\Gamma(a_j)}]\ I_{1,p}^{(a_k-1),(b_{q-p+k}-a_k)}\left\{{}_0F_{q-p}\left((b_l)_1^{q-p}; x\right)\right\}.
\end{aligned}
\tag{49}
$$

This intermediate differintegral relation combined with (47) (for $m = q - p$) gives the following particular form of the general proposition in the case $p < q$.

THEOREM 5.1. *The ${}_pF_q$-function, $p < q$ is a generalized q-tuple fractional (differ)integral of* $\cos_{q-p+1}(x)$, *namely:*

$$
\begin{aligned}
&{}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -x) \\
&= A I_{1,q}^{(\gamma_k),(\delta_k)}\left\{\cos_{q-p+1}\left((q-p+1)x^{\frac{1}{q-p+1}}\right)\right\},
\end{aligned}
\tag{50}
$$

with $A = \sqrt{q-p+1/(2\pi)^{q-p}} \left[\prod_{j=1}^{q} \Gamma(b_j) / \prod_{i=1}^{p} \Gamma(a_i) \right]$ *and parameters* γ_k, δ_k*:*

$$\gamma_k = \begin{cases} \frac{k}{q-p+1} - 1, \\ a_{k-q+p} - 1, \end{cases} ; \delta_k = \begin{cases} b_k - \frac{k}{q-p+1}, & \text{for } k = 1, \dots, q-p \\ b_k - a_{k-q+p}, & \text{for } k = q-p+1, \dots, q. \end{cases} \tag{51}$$

If the conditions

$$b_k > \frac{k}{q-p+1}, \; k = 1, \dots, q-p; \;\; b_{q-p+k} > a_k > 0, \; k = 1, \dots, p \tag{52}$$

are satisfied, then relation (50) gives Poisson type integral representations:

$$\begin{aligned} & {}_pF_q\left(a_1, \dots, a_p; b_1, \dots, b_q; -x\right) \\ & = A \int_0^1 G_{q,q}^{q,0} \left[\sigma \left| \begin{matrix} (b_k)_{k=1}^{q} \\ \frac{k}{q-p+1}{}_{k=1}^{q-p}, \left(a_{k-q+p}\right)_{k=q-p+1}^{q} \end{matrix} \right. \right] \\ & \times \sigma^{-1} \cos_{q-p+1} \left[(q-p+1)(x\sigma)^{\frac{1}{q-p+1}} \right] d\sigma \\ & = A \int_0^1 \underset{(q)}{\dots} \int_0^1 \prod_{k=1}^{q-p} \left[\frac{(1-t_k)^{b_k - \frac{k}{q-p+1} - 1}}{\Gamma\left(b_k - \frac{k}{q-p+1}\right)} \right. \\ & \times \left. t_k^{\frac{k}{q-p+1} - 1} \right] \prod_{k=q-p+1}^{q} \left[\frac{(1-t_k)^{b_k - a_{k-q+p} - 1}}{\Gamma(b_k - a_{k-q+p})} t_k^{a_{k-q+p} - 1} \right] \\ & \times \cos_{q-p+1} \left[(q-p+1)(x t_1 \dots t_q)^{\frac{1}{q-p+1}} \right] dt_1 \dots t_q. \end{aligned} \tag{53}$$

If (52) are not true at least for one of the indices k, then $I_{1,q}^{(\gamma_k),(\delta_k)}$ *in (50) is considered as a generalized fractional derivative of form (42).*

Relation (50) including integral representations (47),(53) and their differintegral analogues, expresses the ${}_pF_q$-functions, $p < q$ as fractional differintegrals (integrals or derivatives) of the generalized cosine function. This fact generalizes the well-known representations like (46),(48) of the Bessel functions via the cosine and thus, suggests the name *Bessel type g.h.f-s* for the functions ${}_pF_q$ with $p < q$. The Bessel function itself is the simplest special function of this *class* of g.h.f-s.

2nd case: $p = q$

Applying relation (43) to a function ${}_pF_p$ consequently $(p-1)$ times, we reach to a ${}_1F_1$-function which on its side is representable as an Erdélyi-Kober operator of the elementary function $x^\alpha \exp x$:

$$\frac{\Gamma(a)}{\Gamma(b)}\, {}_1F_1(a;b;x) = x^{1-a} I_1^{0,b-a}\{x^{a-1}\ e^x\} = \begin{cases} I_1^{a-1,b-a}\{e^x\} & \text{if } b > a \\ D_1^{b-1,a-b}\{e^x\} & \text{if } b < a. \end{cases} \tag{54}$$

Thus, combining (43) and (54), by p steps we obtain the form of the general proposition in this case.

THEOREM 5.2. *If $p = q$, the g.h.f. ${}_pF_p(x)$ is an p-tuple fractional integral or derivative of the elementary function $\{x^{a_1-1}e^x\}$, namely:*

$$ {}_pF_p(a_1,...,a_p;b_1,...,b_p;x) = \Gamma' x^{1-a_1} I_{1,p}^{(\gamma_k),(\delta_k)}\left\{x^{a_1-1}e^x\right\}, \tag{55}$$

where $\gamma_k = a_k - a_1$, $\delta_k = b_k - a_k$, $k = 1,\ldots,p$ and $\Gamma' = \prod\limits_{j=1}^{p}[\Gamma(b_j)/\Gamma(a_j)]$. If

$$b_k > a_k > 0, \quad k = 1,...,p, \tag{56}$$

this relation yields the following integral representations:

$$ {}_pF_p(a_1,\ldots,a_p;\ b_1,\ldots,b_q;x) = \Gamma' \int\limits_0^1 G_{p,p}^{p,0}\left[\sigma \left| \begin{matrix} (b_k)_1^p \\ (a_k)_1^p \end{matrix} \right.\right] \sigma^{-1}\exp(x\sigma)d\sigma$$

$$= \Gamma' \int\limits_0^1 \underset{(p)}{\ldots} \int\limits_0^1 \prod_{k=1}^{p} \left[\frac{(1-t_k)^{b_k-a_k-1} t_k^{a_k-1}}{\Gamma(b_k-a_k)}\right] \exp(xt_1\ldots t_p)dt_1\ldots dt_p. \tag{57}$$

The above theorem justifies separating of all the g.h.f-s ${}_pF_p$ in a *class of g.h.f-s of confluent type*, involving the confluent hypergeometric function ${}_1F_1(a;b;x) = \Phi(a;b;x)$ as a simplest case.

For parameters not satisfying (56), relation (55) gives differintegral expressions. For example, we can introduce *"spherical" g.h.f-s of confluent type* (by analogy with (48)) representable by pure differential operators of $\exp x$.

COROLLARY . *Let all the differences $a_k - b_k = \eta_k, k = 1, \dots, p$ be nonnegative integers. Then, the "differintegral operator" in (55) turns into a differential operator D_η of integer order $\eta = \eta_1 + \cdots + \eta_k \geq 0$, namely:*

$$
\begin{aligned}
{}_pF_p\left(b_1+\eta_1,\dots,b_p+\eta_p;b_1,\dots,b_p;x\right) &= \left[\prod_{j=1}^{p}\frac{\Gamma(b_j)}{\Gamma(b_j+\eta_j)}\right] \\
\times \left[\prod_{k=1}^{p}\prod_{j=1}^{\eta_k}(x\frac{d}{dx}+b_k+j-1)\right]\{\exp x\} &= Q_p(x)\{\exp x\}\,.
\end{aligned}
\tag{58}
$$

Differential representation (58) gives an example how differential formulas for the "spherical" g.h.f-s introduced in [16] can be used for their explicit calculation, especially in the case $p = q$ in the form $Q_p(x)\{\exp x\}$, where $Q_p(x)$ is a p-degree polynomial.

3rd case: $p = q + 1$

The generalized hypergeometric functions ${}_pF_q$, $p = q + 1$ are said to be *g.h.f-s of Gauss type* (see [16], Ch.4) and are considered for $|x| < 1$. In this case the starting specific result is representation (6) of the *Gauss hypergeometric function*, written as an E.-K. fractional integral:

$$
\begin{aligned}
&\frac{\Gamma(a_1)}{\Gamma(b_1)}\,{}_2F_1(a_1,a_2;b_1;x) \\
&= I_1^{a_2-1,b_1-a_2}\{(1-x)^{-a_1}\} = x^{1-a_2}I_1^{0,b_1-a_2}\{x^{a_2-1}(1-x)^{-a_1}\}
\end{aligned}
\tag{59}
$$

for $b_1 > a_2 > 0$, or as a corresponding fractional derivative if $a_2 > b_1 > 0$.

Since by $(q-1)$ steps (43) a ${}_pF_q$-function reduces to a ${}_2F_1$-function and the composition of fractional differintegrals in (43) and (59) gives a q-tuple integral or derivative, we obtain the third form of the above stated proposition.

THEOREM 5.3. *In the unit disk $|x| < 1$ the g.h.f-s of Gauss type ${}_pF_q$, $p = q + 1$ are q-tuple generalized fractional differintegrals of elementary functions of the form $x^\alpha(1-x)^\beta$, namely:*

$$
\begin{aligned}
&{}_{q+1}F_q\left(a_1,\ldots,a_{q+1};b_1,\ldots,b_q;x\right)\\
&= \Gamma''\ x^{1-a_2} I_{1,q}^{(a_{k+1}-1)_1^q,(b_k-a_{k+1})_1^q}\left\{x^{a_2-1}(1-x)^{-a_1}\right\}. \qquad (60)
\end{aligned}
$$

with $\Gamma'' = \prod_{j=1}^{q}[\Gamma(b_j)/\Gamma(a_{j+1})]$. *This means that for*

$$b_k > a_k > 0, \quad k = 1,\ldots,m \qquad (61)$$

the following Poisson type integral representations hold:

$$
\begin{aligned}
{}_{q+1}F_q\left(\pm x\right) &= \Gamma'' \int_0^1 G_{q,q}^{q,0}\left[\sigma \left| \begin{matrix}(b_k)\\(a_{k+1})\end{matrix}\right.\right]\sigma^{-1}(1\mp x\sigma)^{-a_1}d\sigma\\
&= \left[\prod_{j=1}^{q}\frac{\Gamma(b_j)}{\Gamma(a_{j+1})\Gamma(b_j-a_{j+1})}\right]\int_0^1 \underset{(q)}{\cdots}\int_0^1 \prod_{j=1}^{q}\left[(1-t_k)^{b_k-a_{k+1}-1}t_k^{a_{k+1}-1}\right]\\
&\quad\times(1\mp xt_1\ldots t_q)^{-a_1}dt_1\ldots dt_q. \qquad (62)
\end{aligned}
$$

COROLLARY . *For* $q=1$ *representation (62) coincides with (59), written as the known Euler formula (6) for the Gauss function:*

$$ {}_2F_1\left(a_1,a_2;b_1;x\right) = \frac{\Gamma(b_1)}{\Gamma(a_2)\Gamma(b_1-a_2)}\int_0^1 \frac{(1-t)^{b_1-a_2-1}t^{a_2-1}}{(1-xt)^{a_1}}dt, \qquad (63)$$

valid in $|x|<1$.

This formula *proposes a way for an analytical continuation of* ${}_2F_1(x)$ outside the unit disk to the domain $|\arg(1-x)|<\pi$, where the right-hand side of (63) represents an analytical function of x. For the same reasons, *formulas (62) can serve as analytical extensions of the g.h.f-s* ${}_{q+1}F_q(x), q\geq 1$ *outside* $|x|<1$.

The case with parameters not satisfying condition (61) yields generalized fractional derivatives in (60) and provides also useful corollaries. By analogy with the previous two cases, we introduce the notion *spherical g.h.f-s of Gauss type* when all the differences $a_k - b_k = \eta_k, k=1,\ldots,q$ are

nonnegative integers and ${}_{q+1}F_q(x)$ is representable by a purely differential operator of a function $(1-x)^{\beta}$.

Another interesting case concerns the so-called *hypergeometric polynomials*

$$ {}_{p+1}F_q\left(-n,a_1,\ldots,a_p;b_1,\ldots,b_q;x\right) = \sum_{k=0}^{n} \frac{(-n)_k\,(a_1)_k\cdots\left(a_p\right)_k}{(b_1)_k\cdots\left(b_q\right)_k}\cdot\frac{x^k}{k!} \quad (64) $$

when $p=q$. By taking $a_{q+1}=-n$, $n\geq 0$-integer and $a_k>b_k>0, k=1,\ldots,q$, the fractional derivative form of (60) turns into the *Rodrigues type formula*

$$ \left[\prod_{j=1}^{q}\frac{\Gamma(a_j)}{\Gamma(b_j)}\right]\ {}_{p+1}F_q\left(-n,a_1,\ldots,a_q;b_1,\ldots,b_q;x\right) $$
$$ = D_{1,q}^{(b_k-1),(a_k-b_k)}\left\{(1-x)^n\right\} \quad (65) $$
$$ = x^{1-a_q}D_{1,q}^{(b_k-a_q),(a_k-b_k)}\left\{x^{a_1-1}(1-x)^n\right\} = x^{1-b_q}D^{a_q-b_q}x^{a_p-b_{q-1}} $$
$$ \times D^{a_{p-1}-b_{q-1}}\cdots x^{a_3-b_2}D^{a_2-b_2}x^{a_2-b_1}D^{a_1-b_1}\left\{x^{a_1-1}(1-x)^n\right\}. $$

Some special cases of (65) yield *classical Rodrigues formulas.* For example: $p=q=1$ with $a_1=n+1, b_1=1$ and $x\to\frac{1-x}{2}$ yields the *Rodrigues formula for the Legendre polynomials*:

$$ \begin{aligned} P_n(x) &= (-1)^n\,{}_2F_1\left(-n,n+1;1;\frac{1-x}{2}\right) \\ &= \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[\frac{1-x}{2}^n\frac{1+x}{2}^n\right] = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left\{\left(x^2-1\right)^n\right\}; \end{aligned} \quad (66) $$

and $p=q=2$ with $a_1=n+1, b_1=1, a_2=\zeta, b_2=p$ $(\zeta>p>0)$ gives the *Rodrigues formula for the Rice polynomials*, viz.

$$ \begin{aligned} R_n(x) &= {}_3F_2\left(-n,n+1,\zeta;1,p;x\right) \\ &= \frac{\Gamma(p)}{n!\Gamma(\zeta)}\left[\frac{d^n}{dx^n}x^{1-p}\left(\frac{d}{dx}\right)^{\zeta-p}\right]\left\{x^n(1-x)^n\right\}. \end{aligned} \quad (67) $$

All the results mentioned in this section show that there are, essentially, *three kinds of g.h.f-s* ${}_pF_q$, similar in properties and reducible to the

three elementary functions (36) and to the three "initial"(simplest) g.h.f-s ${}_0F_1$, ${}_1F_1$, ${}_2F_1$. Then, many results for them, including approximation formulas, can be transferred from the simplest functions to the generalized hypergeometric functions, using that in general, the Erdélyi-Kober fractional differintegrals and their compositions preserve the asymptotic behaviour near the origin $x = 0$.

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FRACTALS GENERATED BY MÖBIUS TRANSFORMATIONS AND SOME APPLICATIONS

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Abstract

This paper uses a matriceal technique to study properties of the iterates of Möbius transformations when their coefficients are random variables. The algorithms discussed allow to obtain new fractal structures.

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1 Introduction

We denote the complex plane by C. If $D \subset C$ is a complex domain then a rational map of the form

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0, \quad z = -\frac{d}{c} \notin D,$$

is called a *Möbius transformation*. Using a matriceal representation of f, we shall study some properties of the iterates of f provided $a, b, c, d \in C$ are independent random variables. Applications to branching processes are given. Throughout this paper we will denote by $\mathcal{F}$ the set of all Möbius transformations that is

$$\mathcal{F} = \{z \in D \mapsto \frac{az+b}{cz+d}, \quad ad - bc \neq 0, \quad z \neq -\frac{d}{c}\}.$$

Our paper is organized as follows: Section 2 deals with the iterates of f, in Section 3 a generalization is given, Section 4 shows how it is possible to obtain fractals structures using Mobius transformations and Section presents numerical results.

2 The Iterates of f

Denote by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a 2×2–matrix whose elements are the coefficients of $z \mapsto (az+b)/(cz+d)$. Consider $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\{e_1, e_2\}$ being the canonic basis of C and $Z = \begin{pmatrix} z \\ 1 \end{pmatrix}$, be a column vector. We remark that if $f \in \mathcal{F}$ then

$$f(z) = (e_1^t AZ)/(e_2^t AZ) \tag{2.1}$$

where we denoted by x^t the transpose of x. If on $\mathcal{F}$ we consider the composition then $\mathcal{F}$ becomes an abelian group. Moreover, if $f, g \in \mathcal{F}$ have A and B respectively as associated matrices then

$$(f \circ g)(z) = (e_1^t ABZ)/(e_2^t ABZ)$$

hence $(\mathcal{F}, \circ)$ and $(\mathcal{M}_{2\times 2}, \cdot)$ are isomorphic. This fact allows to write the iterates of a function $f \in \mathcal{F}$ as follows

$$f^{(n)}(z) = (\underbrace{f \circ f \circ \ldots \circ f}_{n \ times})(z) = (e_1^t A^n Z)/(e_2^t A^n Z).$$

In this manner, the study of $f^{(n)}$ is transferred to the study of A^n.

If we denote by $Z_1 = \begin{pmatrix} z_1 \\ 1 \end{pmatrix}$ then the iteration formula may be written as follows

$$z_{n+1} = (e_1^t A^n Z_1)/(e_2^t A^n Z_1), \quad n \in N^*$$

We shall use Perron's formula applied for the n-th power of a $r \times r$-matrix

$$A^n = \sum_{j=1}^{k} \frac{1}{(m_j-1)!} \left[\frac{d^{m_j-1}}{d\lambda^{m_j-1}} \left\{ \lambda^n (\lambda - \lambda_j)^{m_j} (\lambda I - A)^{-1} \right\} \right]_{\lambda=\lambda_j}$$

where $m_1, \ldots, m_k$ are the algebraic multiplicities of eigenvalues $\lambda_1, \ldots, \lambda_k$, $m_1 + \ldots + m_k = r$, $k < r$. In our case, if λ_1, λ_2 are the zeros of the characterictic polynomial of A, $\lambda^2 - (a+d)\lambda + ad - bc = 0$ then

$$A^n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}.$$

If $\lambda_1 = \rho_1 \exp(i\alpha_1)$, $\lambda_2 = \rho_2 \exp(i\alpha_2)$, $\rho_1, \rho_2 \geq 0$, $\alpha_1, \alpha_2 \in [0, 2\pi)$ are the complex eigenvalues of A, then the iterate $f^{(n)}$ may be written as

$$f^{(n)}(z) = \frac{\left[\rho_1^{n+1} E_{n+1}^1 - \rho_2^{n+1} E_{n+1}^2 - d\left(\rho_1^n E_n^1 - \rho_2^n E_n^2\right)\right] z + b\left(\rho_1^n E_n^1 - \rho_2^n E_n^2\right)}{c\left(\rho_1^n E_n^1 - \rho_2^n E_n^2\right) z + \rho_1^{n+1} E_{n+1}^1 - \rho_2^{n+1} E_{n+1}^2 - a\left(\rho_1^n E_n^1 - \rho_2^n E_n^2\right)}$$

where we denoted $E_n^k = \exp(i\alpha_k n)$, $k = 1, 2$.

Taking into account these general backgrounds, we shall study the following cases:

I. $\rho_1 \neq \rho_2$, $\alpha_1 \neq \alpha_2$. Suppose $\rho_1 > \rho_2$ and denote $r_n = \left(\frac{\rho_2}{\rho_1}\right)^n$. After some straightforward calculation we get

$$\lim_{n \to \infty} f^{(n)}(z) = \frac{[\rho_1 \exp(i\alpha_1) - d]\, z + b}{cz + \rho_1 \exp(i\alpha_1) - a}.$$

But λ_1 is eigenvalue of A, therefore

$$\lim_{n \to \infty} f^{(n)}(z) = \frac{\lambda_1 - d}{c} = \frac{b}{\lambda_1 - a}.$$

Hence, independent of z_0 the sequence of iterates $\left\{f^{(n)}(z_0)\right\}_{n \geq 0}$ is convergent to the complex number $(\lambda_1 - d)/c$, $c \neq 0$.

II. $a, b, c, d \in R$. If we denote $\lambda_1 = \rho \exp(i\alpha)$, $\lambda_2 = \rho \exp(-i\alpha)$, with $\rho > 0$ and $\alpha \in [0, 2\pi)$, then

$$f^{(n)}(z) = \frac{[\rho \sin(n+1)\alpha - d \sin n\alpha] z + b \sin n\alpha}{(c \sin n\alpha) z + \rho \sin(n+1)\alpha - a \sin n\alpha}.$$

Suppose that exists $k \in N^*$ such that $\alpha = \frac{\pi}{k}$. If $n = kp$, $p \in N^*$ then $f^{(n)}(z) = z$, that is the identity of $\mathcal{F}$. For $n = kp - 1$ we have $f^{(n)}(z) = (-dz+b)/(cz-a) = f^{-1}(z)$. Therefore, the iterates $\{f^{(n)}(z)\}$ have the form

$$\begin{gathered} f(z), \quad f^{(2)}(z), \ldots, \quad f^{(k-1)}(z) = f^{-1}(z), \quad f^{(k)}(z) = z \\ f^{(k+1)}(z), \quad f^{(k+2)}(z), \ldots, \quad f^{(2k-1)}(z) = f^{-1}(z), \quad f^{(2k)}(z) = z, \ldots \end{gathered}$$

We can deduce that the sequence $z_n = f^{(n)}(z)$, $n = 0, 1, 2, \ldots$ has a finite number of points (cyclic points). These points belong to the circle of equation

$$z\bar{z} + \alpha_1(z+\bar{z}) - i\alpha_2(z-\bar{z}) + \alpha_3 = 0,$$

and $\alpha_1, \alpha_2, \alpha_3 \in R$ are solutions of the following system

$$\begin{pmatrix} z_1 - \bar{z}_1 & i(\bar{z}_1 - z_1) & 1 \\ z_2 - \bar{z}_2 & i(\bar{z}_2 - z_2) & 1 \\ z_3 - \bar{z}_3 & i(\bar{z}_3 - z_3) & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -|z_1|^2 \\ -|z_2|^2 \\ -|z_3|^2 \end{pmatrix} \tag{2.2}$$

where $z_1 = f(z_0)$, $z_2 = f(z_1)$, $z_3 = f(z_2)$ with z_0 fixed. If does not exist k such that $\alpha = \frac{\pi}{k}$, then the iterates $f^{(n)}(z)$ belong to a circle of equation (2.2) and the sequence $\{f^{(n)}(z)\}_{n \geq 0}$ with z_0 fixed is mapped into the points of this circle, [3].

III. $a, b, c, d \in R_+$. According the theorem of Perron-Frobenius, a positive matrix has a dominant positive eigenvalue and the corresponding eigenvector has positive elements. Let's denote this eigenvalue by λ_1 and let us consider $\lambda_2 < \lambda_1$, $\lambda_1, \lambda_2 \in R$. We denote by u_1 (v_1), u_2 (v_2) the corresponding left (right) eigenvectors. From the normality condition $< u_i, v_i >= 1$, $i = 1, 2$ where $<,>$ is the inner product we have

$$A^n = \lambda_1^n v_1 u_1^t + \lambda_2^n v_2 u_2^t.$$

In this case, after some calculations, we get

$$f^{(n)}(z) = \frac{(\lambda_1^n \pi_1 \epsilon_1 + \lambda_2^n \alpha_1 \beta_1)\, z + \lambda_1^n \pi_1 \epsilon_2 + \lambda_2^n \alpha_1 \beta_2}{(\lambda_1^n \pi_2 \epsilon_1 + \lambda_2^n \alpha_2 \beta_1)\, z + \lambda_1^n \pi_2 \epsilon_2 + \lambda_2^n \alpha_2 \beta_2}.$$

Here we denoted $u_1 = \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}$, $u_2 = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$, $v_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$, $v_2 = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$, and $\pi_i > 0$, $\epsilon_i > 0$, $i = 1, 2$ satisfy $\pi_1\epsilon_1 + \pi_2\epsilon_2 = 1$. Dividing by λ_1^n and passing to limit we have

$$\lim_{n \to \infty} f^{(n)}(z) = \frac{\pi_1}{\pi_2}$$

hence in this case, independent of z_0, the iterates are convergent to a point located on the real axis.

Application. Let's consider a branching process, [1], $\{\xi_n, n \in N\}$ whose generating function is given by

$$f(z) = \frac{1-b-c}{1-c} + \frac{bz}{1-cz}, \quad 0 < b, c < 1, \; b + c \leq 1.$$

$f(z)$ is the sum of power series $\sum_{k\geq 1} p_k z^k$, with $p_k = bc^{k-1}$, $k = 1, 2, \ldots$ and $p_0 = \frac{1-b-c}{1-c}$. In a branching process, ξ_n represents the number of individuals from a given population at $n-$th generation and the expectation will be

$$m = f'(1) = \sum_{k\geq 1} kp_k = \frac{b}{(1-c)^2}.$$

The matrix of f will be

$$A = \begin{pmatrix} b + c^2 - c & 1 - b - c \\ c^2 - c & 1 - c \end{pmatrix}$$

and its eigenvalues are $\lambda_1 = b$, $\lambda_2 = (1-c)^2$.

From (2.4) we obtain

$$\lim_{n\to\infty} f^{(n)}(z) = \frac{\lambda_M + b - 1}{c - 1},$$

where $\lambda_M = \max\{\lambda_1, \lambda_2\}$. If $m > 1$ then $f^{(n)}(z) \to \frac{1-b-c}{c-c^2}$ while if $m < 1$ then $f^{(n)}(z) \to 1$. The limit of $f^{(n)}(z)$ represents the probability of extinction of the process, [1].

3 Generalization

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_r\}$ be a set of r $2 \times 2-$ non- singular matrices which correspond to some Möbius transformations form $\mathcal{F}$. If $A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathcal{A}$, $n \geq 1$ then we build the sequence as follows:

$$z_0 \text{ given}, \quad z_1 = f_1(z_0), \quad z_2 = (f_2 \circ f_1)(z_0), \ldots.$$

Taking into account (2.1) we obtain the matriceal representation of this sequence as

$$z_{n+1} = \frac{e_1^t\,(A_n \ldots A_1)\,Z}{e_2^t\,(A_n \ldots A_1)\,Z}, \quad Z = \begin{pmatrix} z_0 \\ 1 \end{pmatrix}.$$

Let us consider $W = C$, $X = \mathcal{A}$, $\mathcal{W}$ the borelians of W, and $\mathcal{X}$ the set of parts of X. Consider a probability measure $\{\rho_x(w),\ w \in W\}_{x\in X}$ with $\rho_x(w) = \rho_x > 0$ and $\sum_{x\in X} \rho_x(w) = 1$. Moreover, for all $w \in W$ we consider the family of applications $f\colon W \times X \to W$ which are $(\mathcal{W} \otimes \mathcal{X}, \mathcal{W})-$measurable, $f \in \mathcal{F}$. The

tuple $\{(W,\mathcal{W}),(X,\mathcal{X}),f,\rho\}$ is called random system with complete connections, [4].

We remark that the sequence obtained above, $(z_n)_{n\geq 0}$, where for each n a random $f_n(z)$ is choosen, defines a Markov chain. The transition probability of this Markov chain is

$$P(z,A)=\sum_{x\in X} I_A(f_x(z))\rho_x \tag{3.1}$$

for all $z\in W$ and $A\in\mathcal{W}$. This is the Markov chain attached to the random system with complete connections.

Let us suppose without loss of generality that the matrices A_k are positive (the general case is similar). For practical reasons we have to study the asymptotic behaviour of the following random product of matrices

$$A^{(n)}=A_n\dots A_2A_1.$$

For this purpose, we consider $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ a positive matrix and its decomposition as a product of a diagonal matrix and a stochastic one as follows:

$$D=\begin{pmatrix} a+b & 0\\ 0 & c+d\end{pmatrix},\quad P=\begin{pmatrix} \frac{a}{a+b} & \frac{b}{a+b}\\ \frac{c}{c+d} & \frac{d}{c+d}\end{pmatrix}.$$

Therefore $A=D\cdot P$. Using this decomposition for $A^{(n)}$ we have succesively

$$\begin{aligned}A^{(n)}&=D_nP_nA_{n-1}\dots A_1=D_n\bar{A}_{n-1}A_{n-2}\dots A_1\\ &=D_nD_{n-1}P_{n-2}A_{n-3}\dots A_1=\dots=D_1\dots D_nP_n.\end{aligned}$$

Here $D_1,\dots,D_n$ are diagonal matrices, P_n is a stochastic one and $\bar{A}_{n-1}$ P_nA_{n-1}. In general, if we denote by $\bar{A}$ the product between a general matrix and a stochastic one that is $\bar{A}=AP$ from the componentwise interpretation we get

$$\min_{i,j} a_{ij}<\min_{i,j}\bar{a}_{ij}<\max_{i,j}\bar{a}_{ij}<\max_{i,j} a_{ij}. \tag{3.2}$$

If we denote

$$A^{(n)}=\begin{pmatrix} a^{(n)} & b^{(n)}\\ c^{(n)} & d^{(n)}\end{pmatrix},\quad D_1\cdots D_n=\begin{pmatrix} h^{(n)} & 0\\ 0 & g^{(n)}\end{pmatrix},$$
$$P_n=\begin{pmatrix} p_{1n} & 1-p_{1n}\\ 1-p_{2n} & p_{2n}\end{pmatrix}$$

then we have

$$A^{(n)}=\begin{pmatrix} h^{(n)}p_{1n} & h^{(n)}(1-p_{1n})\\ g^{(n)}(1-p_{2n}) & g^{(n)}p_{2n}\end{pmatrix}.$$

Based on the inequalities (3.2) and taking into account the positivity of A_n, it results that exists $\delta>0$ and $n_0>1$ such that $p_{1n}>\delta,\ p_{2n}>\delta$ for all $n>n_0$.

3.1 Proposition. *In the conditions described above, there exists and it is finite the limit*

$$\lim_{n\to\infty} \frac{1}{n} \log a^{(n)}.$$

Sketch of proof. Indeed, we have

$$\begin{aligned}\tfrac{1}{n}\log a^{(n)} &= \tfrac{1}{n}\log h^{(n)} p_{1n} = \tfrac{1}{n}\log h^{(n)} + \tfrac{1}{n}\log p_{1n}\\ &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\log h_i + \tfrac{1}{n}\log p_{1n}\end{aligned}$$

where $h^{(n)} = h_1 \cdots h_n$ with $h_i > 0$ for every $i = 1, 2, \ldots$ But for n large enough $p_{1n} > 0$ and then $\frac{1}{n}\log p_{1n} \to 0$. From the law of large numbers it results that $\lim \frac{1}{n}\sum_{i=1}^{n}\log h_i$ exists and it is finite. Let's denote this limit by h. Therefore, $\frac{1}{n}\log a^{(n)} \to h$. For the other elements of $A^{(n)}$ the demonstration is similar.

By Proposition 3.1 it results that if n is large enough then $A^{(n)}$ approaches to a matrix with proportional rows $A^{(n)} \approx \begin{pmatrix} e^{nh} & e^{nh} \\ e^{ng} & e^{ng} \end{pmatrix}$, where h and g are positive random variables. Therefore, the sequence $(z_n)_{n\geq 0}$ is given by the recurrence

$$z_{n+1} \approx \frac{e^{nh}z + e^{nh}}{e^{ng}z + e^{ng}} = e^{n(h-g)}$$

hence independent of z_0.

4 Fractals from Möbius Transformations

Let (C, D) be the complex metric space endowed with the distance $d(z_1, z_2) = |z_1 - z_2|$, $z_1, z_2 \in C$ and $\{f_1, \ldots, f_r\}$ Möbius transformations from $\mathcal{F}$ whose matrices are $A_1, \ldots, A_r$ respectively. Let us consider also the set

$$\{\rho_x, x \in X\} \quad X = \{1, 2, \ldots, r\}, \quad \rho_x > 0, \quad \sum_{x\in X} \rho_x = 1.$$

It is obviuos that for all $k \in \{1, 2, \ldots, r\}$, f_k are Lipschitzian complex functions defined on C, that is there exist s_k such that for all $z, w \in C$ the inequalities hold $d(f_k(z), f_k(w)) \leq s_k d(z, w)$. Barnsley, [2], proves that the Markov chain $(z_n)_{n\geq 0}$ built in Section 3 having the transition probabilities (3.1) and the transition operator

$$(Tg)(z) = \int g(y)P(z, dy) = \sum_{k=1}^{r} \rho_k (c \circ f_k)(z)$$

has a unique invariant measure without the conditions that f_k would be contractions and the space of states would be compact.

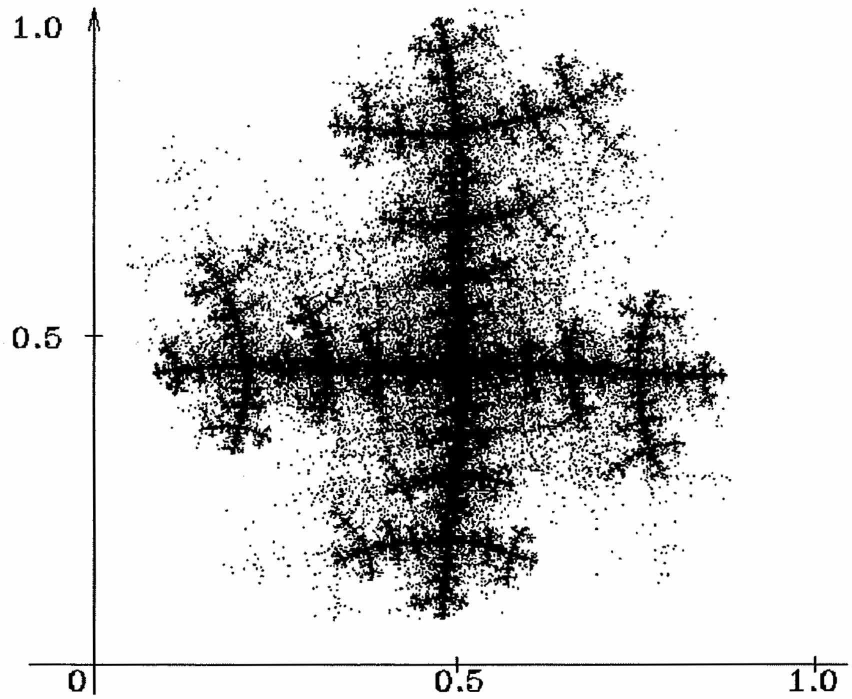

Fig. 1

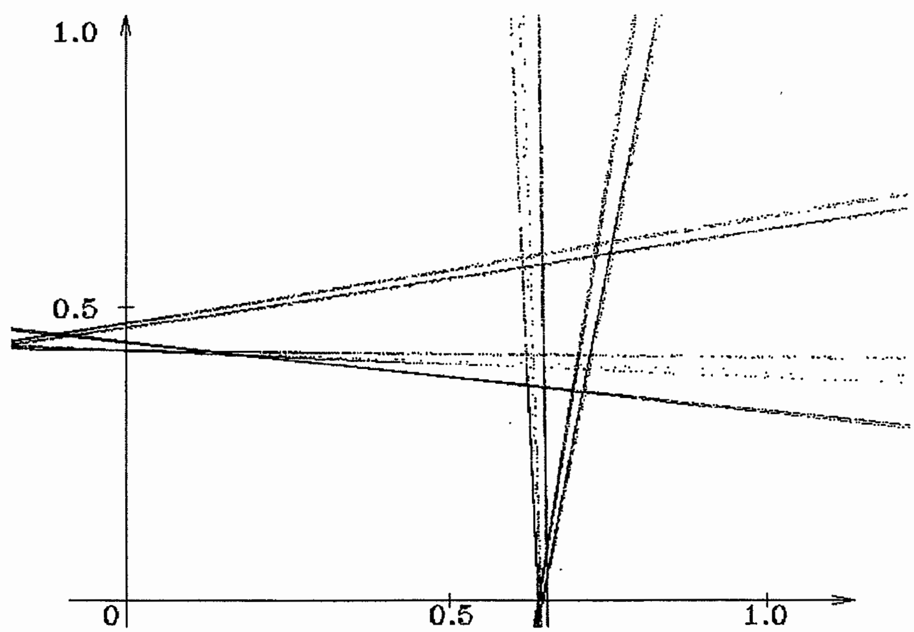

Fig. 2

4.1 Theorem. [2] *Suppose that there exists* $0 < r < 1$ *such that*

$$\prod_{k=1}^{r} \left[\frac{d(f_k(w), f_k(z))}{d(w,z)} \right]^{\rho_k} \leq r \tag{4.1}$$

uniform with respect to $w, z \in Y$, Y *being a complete metric space. Then, the Markov chain* $(z_n)_{n \geq 0}$, *built as above, admits a unique invariant measure* μ.

The support of μ is a fractal. In our case, the states space is C and for a set of Möbius transformations $f_1, \ldots, f_r$ there exist $\rho_1, \ldots, \rho_r$, $\sum_{k=1}^{r} \rho_k = 1$, $\rho_k > 0$ such that (4.1) holds.

5 Numerical Examples

First, we remark that any Möbius transformation may be written as

$$f(z) = \frac{\alpha z + \beta}{z + \gamma}, \quad \alpha, \beta, \gamma \in C.$$

It generates a random number q. If $q \in (0, 0.8)$ then $\alpha = -3 + 5i$, $\beta = 0.8 - i$, and $\gamma = -4 + 5i$. If not, for $q \in [0.8, 0.86)$ choose $\alpha = 2 + 0.4i$, $\beta = -0.7 - i$, $\gamma = 1 - 2i$ else $\alpha = -4 + 4i$, $\beta = 0.7 - i$, $\gamma = 5 + 2i$. Compute and plot a random orbit of z_0 under f. In Fig. 1 is displayed the result after 150,000 iterations.

A special case of iterative relation with applications to n-dimensional branching processes is $x_{k+1} = f(x_k)$, $k \geq 0$, $x_k \in R^n$. Here $f(x) = (\frac{f_1(x)}{g(x)}, \ldots \frac{f_n(x)}{g(x)})$ has the components of the form

$$f_i(x) = \frac{a_i^1 x_1 + a_i^2 x_2 + \cdots + a_i^n x_n + a_i^0}{b_1 x_1 + b_2 x_2 + \cdots + b_n x^n + b_0}, \quad i = 1, 2, \ldots, n, \quad x = (x_1, \ldots, x_n).$$

Ulam's conjecture, [5], states that the points x_k belong to a cone in R^n. For $n = 2$, we have the following iterative relation

$$x_{k+1} = \frac{a_1^1 x_k + a_2^1 y_k + a_0^1}{b_1 x_k + b_2 y_k + b_0}, \quad y_{k+1} = \frac{a_1^2 x_k + a_2^2 y_k + a_0^2}{b_1 x_k + b_2 y_k + b_0} \tag{5.1}$$

If we choose two iterative relations of the form (5.1) which succeed each other with probablilities p and $1 - p$, $0 < p < 1$ respectively then we find a Markov chain. For 150,000 iterations, the suppurt of the invariant measure of this Markov chain has the appearance given in Fig. 2. Here we have: with $0 < p < 0.5$

$$x_{k+1} = \frac{-2x_k + 3y_k - 1}{2.1x_k + 3.2y_k + 3}, \quad y_{k+1} = \frac{x_k + 2y_k + 1}{2.1x_k + 3.2y_k + 3}$$

else

$$x_{k+1} = \frac{1.5x_k + 2.1y_k + 1.2}{2.4x_k + 2.8y_k + 2}, \quad y_{k+1} = \frac{2x_k + 1.6y_k - 2}{2.4x_k + 2.8y_k + 2}.$$

So, Ulam's conjecture is confirmed.

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A SERIES-ITERATION METHOD IN THE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

The main contribution of this paper is the treatmnent of the theory of linear ordinary differential equations by the well-known traditional methods of series and iterations. In the iteration method, instead of the continuity we require the analyticity of the coefficients of the differential equations. Then the iterations can be carried out for the arbitrary form of the coefficients. Our procedure determines the fundamental system of particular integrals $\{y_1, y_2, \dots, y_n\}$ with arbitrary precision. Therefore our theory of differential equations is non-Liouville. (In the Liouville theory we know only that the solution has the form $y(x) = \sum_{k=0}^{n} C_k y_k$, without a procedure for finding the particular integrals). The Wronskian $W(y_1, y_2, \dots, y_n) = 1!2! \cdots (n-1)!$ of the fundamental system of a differential equation of n-th order has a new meaning. It becomes the basic metric relation in the space of particular solutions. Knowing that linear differential equations of the second order with constant coefficients correspond to usual trigonometry there exists a natural generalization of the trigonometry (elliptical and hyperbolical). From the Wronskian as a basic trigonometrical relation the Euler formula, addition formulae etc. follow. The solutions in the form of a series of multiple integrals provide a natural approach to approximate calculations. The approximate solutions are especially suitable in the case of monotone coefficients of differential equations. Working in the space of the fundamental particular integrals $y_1 \otimes y_2 \otimes \cdots \otimes y_n$ one can obtain interesting geometrical relations.

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1. The Basic Principle and Definitions

The expression

$$y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_n(x)y = b(x), \tag{1.1}$$

where $a_1(x)$, $a_2(x)$, $\cdots$, $a_n(x)$, $b(x)$ are analytical coefficients is called linear ordinary differential equation of n-th order.

In the theory of differential equations the following well known principle is valid:

The solution of a differential equation depends of the coefficients on the equation only.

This principle cannot be proved because it is not possible, for a lot of differential equations, to express the solution via the coefficients. Some equations are not solvable, some have multiple solutions, while of some is known that these have solutions but the algorithm for their construction do not exist (only the theorem of existence). However, practical experience during the history of differential equations shows that the above principle is valid.

This paper is excellent "proof" of this principle. We will to try to show that all types of solution (general, particular ...) depend uniquely on the coefficients of the equation.

Definition. The solution of differential equation (1.1) is called a **general quadrature** if it has the form of a series which can be expressed by the coefficients of the equation (1.1).

We now enumerate three different possibilities:

a) The solution has a closed form in terms of well-known functions,

b) The solution has the form of a series, not of x but of the coefficients of the equation (1.1),

c) The solution is a power series where the coefficients depend on the coefficients (1.1) explicitly.

We will call all these possibilities quadrature solutions in the wider sense.

2. Linear Equations of I Order

First we consider the simplest possible case, the linear canonical differential equation of the first order

$$y' + a(x)y = 0, \tag{2.1}$$

with the intention of explaining the series-iteration method.

The first step in our consideration is to integrate differential equation (2.1) and get

$$y = -\int_0^x a(x)y\,dx + C, \tag{2.2}$$

where C is the constant of integration. The next step is to substitute the all right hand side of the equation (2.2) instead of the variable y in the right hand sade of the equation (2.2)

$$y = -\int_0^x a(x)\left[-\int_0^x a(x)y\,dx + C\right]dx + C, \tag{2.3}$$

and we obtain

$$y = \int_0^x a(x)\,dx \int_0^x a(x)\,y\,dx - C\int_0^x a(x)\,dx + C. \tag{2.4}$$

The next step is to substitute the variable y from (2.2) in the right hand sade of the equation (2.4). If we continue this process of iteration we obtain the following form of solution of the differential equation (2.1)

$$\begin{aligned} y =& C\left[1 - \int_0^x a(x)\,dx + \int_0^x a(x)\,dx \int_0^x a(x)\,dx\right.\\ &\left. - \int_0^x a(x)\,dx \int_0^x a(x)\,dx \int_0^x a(x)\,dx + \cdots\right]. \end{aligned} \tag{2.5}$$

Using the identity (Dyson formula [1])

$$\underbrace{\int_0^x a(x)\,dx \int_0^x a(x)\,dx \cdots \int_0^x a(x)\,dx}_{k} = \frac{1}{k!}\left(\int_0^x a(x)\,dx\right)^k, \tag{2.6}$$

the solution (2.5) can be transformed into

$$y = \sum_{k=0}^{\infty} \frac{1}{k!} \left(- \int_0^x a(x)\, dx \right)^k , \tag{2.7}$$

which is, of course, the series of the exponential function, i.e.

$$y = Ce^{-\int_0^x a(x)\, dx}. \tag{2.8}$$

Similarly, the nonhomogeneous differential equation of the first order

$$y' + a(x)y = b(x), \tag{2.9}$$

has a first integral

$$y = -\int_0^x a(x)y\, dx + \int_0^x b(x)\, dx + C. \tag{2.10}$$

Again if we substitute the all right hand side of the equation (2.10) instead of the variable y in the right hand sade of the equation (2.10) we find

$$\begin{aligned} y &= -\int_0^x a(x) \left[-\int_0^x a(x)y\, dx + \int_0^x b(x) + C \right] dx + \int_0^x b(x)\, dx + C \\ &= \int_0^x a(x)\, dx \int_0^x a(x)y\, dx - \int_0^x a(x)\, dx \int_0^x b(x)\, dx - C\int_0^x a(x)\, dx + \int_0^x b(x)\, dx + C. \end{aligned} \tag{2.11}$$

Continuing step by step the process of iteration we obtain the solution of nonhomogeneous differential equation (2.9) in the form

$$y = Ce^{-\int_0^x a(x)\, dx} + Y_p, \tag{2.12}$$

where Y_p is the particular integral

$$Y_p = \int_0^x b(x)\, dx - \int_0^x a(x)\, dx \int_0^x a(x)\, dx + \int_0^x a(x)\, dx \int_0^x a(x)\, dx \int_0^x b(x)\, dx - \cdots \tag{2.13}$$

After differentiation of the expression (2.13) we have

$$Y_p{}' = b(x) - a(x)\left[\int_0^x b(x)\,dx - \int_0^x a(x)\,dx \int_0^x b(x)\,dx + \cdots\right], \qquad (2.14)$$

So, the particular integral (2.13) satisfies the identity

$$Y_p{}' + a(x)Y_p = b(x). \qquad (2.15)$$

Because the identity (2.15) is the same as a differential equation (2.9), the particular sulution Y_p is equal to well-known particular solution of the differential equation (2.9)

$$Y_p = e^{-\int_0^x a(x)\,dx} \int_0^x b(x) e^{\int_0^x a(x)\,dx}\,dx. \qquad (2.16)$$

The main attraction of the method of iteration is that in the formula of solution (2.12) appears naturaly the solution of homogeneous part (with a constant of integration) an also the particular integral.

3. Linear Equations of II Order

Let us study the canonical differential equation of the second order with one coefficient [2-5]

$$y'' + a(x)y = 0, \qquad (3.1)$$

where $a(x)$ is an analytical positive function in the interval $[x_1, x_2]$. Equation (3.1) is well known and with a special form of $a(x)$ it becomes the equation of Mathieu, Lamé, Hill, Bessel etc. In classical mechanics equation (3.1) represents harmonic oscillator with time-dependence mass. Also this type of equations is equivalent to stationary Schrödinger equation which plays a fundamental role in quantum mechanics.

Now we applied the method of iterations on the differential equation (3.1). After the integration we have

$$
\begin{aligned}
y' &= -\int_0^x a(x)y\,dx + C_2, \\
y &= -\int_0^x dx \int_0^x a(x)y\,dx + C_2 x + C_1,
\end{aligned}
\tag{3.2}
$$

where C_1 and C_2 are the constants of integration. Repeating the procedure in the previous paragraph we substitute the all right hand side of the last equation instead the variable y in the right hand side of the same equation and we find

$$
\begin{aligned}
y = &\int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)y\,dx^2 - C_2 \int_0^x\int_0^x xa(x)\,dx^2 \\
&- \int_0^x\int_0^x a(x)\,dx^2 + C_2 x + C_1,
\end{aligned}
\tag{3.3}
$$

where we denoted

$$
\int_0^x\int_0^x f(x)\,dx^2 \equiv \int_0^x dx \int_0^x f(x)\,dx. \tag{3.4}
$$

If we continue the process of iteration we obtain

$$
\begin{aligned}
y = C_1 &\left[1 - \int_0^x\int_0^x a(x)\,dx^2 + \int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 - \cdots\right] \\
+ C_2 &\left[x - \int_0^x\int_0^x xa(x)\,dx^2 + \int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x xa(x)\,dx^2 - \cdots\right].
\end{aligned}
\tag{3.5}
$$

So, the solution of the differntial equation (3.1) has the form of summa of two series of double integrals

$$
y(x) = C_1 y_1 + C_2 y_2, \tag{3.6}
$$

where

$$
y_1 = \sum_{k=0}^{\infty} (-1)^k \underbrace{\int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 \cdots \int_0^x\int_0^x a(x)\,dx^2}_{k \text{ double integrals}}, \tag{3.7}
$$

$$y_2 = x + \sum_{k=1}^{\infty}(-1)^k \underbrace{\int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 \cdots \int_0^x\int_0^x x a(x)\,dx^2}_{k \text{ double integrals}}. \tag{3.8}$$

The form of solution as a summa of multiple integrals is good possibility to use some software for analytical integration. In the reference [6] we have used the Mathematica [7] to perform integrations in the case of $a(x) = \{e^x, \ln x, \cos x, \sin x\}$.

Theorem. Let $a(x)$ be a analytic coefficient (not necessarily positive) in the interval $[0, X]$, then the series (3.7) and (3.8) converge uniformly for all $x \in [0, X]$.

Proof. If $a(x)$ is an analytic function it is continuous and bounded. So there exists the number M such that $|a(x)| < M$. Then we have the majoranta

$$\begin{aligned}
|y_1| &< 1 + \int_0^x\int_0^x |a(x)|\,dx^2 + \int_0^x\int_0^x |a(x)|\,dx^2 \int_0^x\int_0^x |a(x)|\,dx^2 + \cdots \\
&< 1 + M\int_0^x\int_0^x dx^2 + M^2 \int_0^x\int_0^x dx^2 \int_0^x\int_0^x dx^2 + \cdots \\
&= 1 + M\frac{x^2}{2!} + M^2\frac{x^4}{4!} + \cdots + M^n\frac{x^{2n}}{(2n)!} + \cdots
\end{aligned} \tag{3.9}$$

or

$$|y_1| < \sum_{n=0}^{\infty} \frac{\left(\sqrt{M}x\right)^{2n}}{n!}. \tag{3.10}$$

Using the D'Alamberts ratio test, for the n-th term in the series (3.10) is valid

$$\lim_{n\to\infty}\frac{A_{n+1}}{A_n} = \lim_{n\to\infty}\frac{\frac{\left(\sqrt{M}x\right)^{2(n+1)}}{(n+1)!}}{\frac{\left(\sqrt{M}x\right)^{2n}}{n!}} = \lim_{n\to\infty}\frac{\left(\sqrt{M}x\right)^2}{n+1} = 0. \tag{3.11}$$

Then the series (3.10) converges for any x. Therefore, the smaller series (3.7) with positive and negative terms converges uniformly for any analytical coefficient $a(x)$ QED.

Similarly, for the series (3.8) we can write

$$\begin{aligned}|y_2| &< |x| + \int_0^x\int_0^x |xa(x)|\,dx^2 + \int_0^x\int_0^x |a(x)|\,dx^2 \int_0^x\int_0^x |xa(x)|\,dx^2 + \cdots \\ &< |x| + M\int_0^x\int_0^x |x|\,dx^2 + M^2\int_0^x\int_0^x |x|\,dx^2 \int_0^x\int_0^x |x|\,dx^2 + \cdots \\ &= |x| + M\frac{|x|^2}{3!} + M^2\frac{|x|^5}{5!} + \cdots\end{aligned} \tag{3.12}$$

or

$$|y_2| < \sum_{n=0}^{\infty} M^n \frac{|x|^{2n+1}}{(2n+1)!}. \tag{3.13}$$

Because the last majoring series, according to the D'Alamberts test, converges, the series (3.8) converges uniformly.

The consequence of the previous theorem is that in the series (3.7) and (3.8) the usual analytical operation of such differentiation and integration term-by-term can be applied.

Let us consider the homogeneous differentail equation

$$y'' + a(x)y' + b(x)y = 0, \tag{3.14}$$

where $a(x)$ and $b(x)$ are analytical functions. With substitution

$$y = e^{-\frac{1}{2}\int_0^x a(x)dx} \cdot z \tag{3.15}$$

the equation (3.14) can be transformed into a canonical type

$$z'' + A(x)z = 0, \tag{3.16}$$

where

$$A(x) = b(x) - \frac{1}{4}a^2(x) - \frac{a'(x)}{2}. \tag{3.17}$$

Let us we study the most general case of the differential equation of the II order, so called the nonhomogeneous equation

$$y'' + a(x)y' + b(x)y = f(x), \tag{3.18}$$

where $a(x)$, $b(x)$ and $f(x)$ an analytical coefficients. Then by the substitution (3.15) the above differential equation transforms into

$$z'' + A(x)z(x) = f_1(x), \tag{3.19}$$

where $A(x)$ is determined by the formula (3.17) while

$$f_1(x) = e^{\frac{1}{2}\int_0^x a(x)dx} f(x). \tag{3.20}$$

So, instead the equation (3.18) we can to study the equivalent equation (3.19).

Theorem. The differential equation (3.19) has a particular solution

$$\begin{aligned} Y_p = &\int_0^x\int_0^x f_1(x)\,dx^2 - \int_0^x\int_0^x \left(\int_0^x\int_0^x f_1(x)\,dx^2\right) A(x)\,dx^2 \\ &+ \int_0^x\int_0^x A(x)\,dx^2 \int_0^x\int_0^x \left(\int_0^x\int_0^x f_1(x)\,dx^2\right) A(x)\,dx^2 - \cdots . \end{aligned} \tag{3.21}$$

The proof follows directly by differentiation of the expression (3.21) and substitution into the equation (3.19).

Theorem. The general solution (in wider sence) of nonhomogeneous differential equation of the II order (3.14) has the following form

$$\begin{aligned} y = e^{-\frac{1}{2}\int_0^x a(x)dx} &\left\{ C_1 \left[1 - \int_0^x\int_0^x A(x)\,dx^2 + \int_0^x\int_0^x A(x)\,dx^2 \int_0^x\int_0^x A(x)\,dx^2 - \cdots\right] \right. \\ &+ C_2 \left[x - \int_0^x\int_0^x xA(x)\,dx^2 + \int_0^x\int_0^x A(x)\,dx^2 \int_0^x\int_0^x xA(x)\,dx^2 - \cdots\right] \\ &+ \int_0^x\int_0^x f_1(x)e^{\frac{1}{2}\int_0^x a(x)dx}\,dx^2 - \int_0^x\int_0^x A(x)\left(f_1(x)e^{\frac{1}{2}\int_0^x a(x)dx}\right) dx^2 \\ &\left. + \int_0^x\int_0^x A(x)\,dx^2 \int_0^x\int_0^x A(x)\,dx^2 \left(f_1(x)e^{\frac{1}{2}\int_0^x a(x)dx}\right) dx^2 - \cdots \right\}. \end{aligned} \tag{3.22}$$

4. Generalized Trigonometry of Canonical Equations

In differential equation (3.1) by putting $a(x) = 1$ we obtain the equation of a harmonic oscillator. Then the solution (3.7) reduces to

$$\begin{aligned} y_1 &= 1 - \int_0^x\int_0^x 1 \cdot dx^2 + \int_0^x\int_0^x 1 \cdot dx^2 \int_0^x\int_0^x 1 \cdot dx^2 - \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned} \tag{4.1}$$

and it is standard cosine function

$$y_1 = \cos x = \cos_{a(x)=1} x. \tag{4.2}$$

And analogously

$$\begin{aligned} y_2 &= x - \int_0^x\int_0^x 1 \cdot x dx^2 + \int_0^x\int_0^x 1 \cdot dx^2 \int_0^x\int_0^x 1 \cdot x dx^2 - \cdots \\ &= 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned} \tag{4.3}$$

which denotes sine in usual trigonometry

$$y_2 = \sin x = \sin_{a(x)=1} x. \tag{4.4}$$

Generalizing the above facts [3] we make the following

Definition. The generalized cosine is the solution of the differential equation (3.1) ($a(x) > 0$) with the initial conditions

$$y(0) = 1 \qquad \text{and} \qquad y'(0) = 0. \tag{4.5}$$

Therefore

$$y_1(x) \equiv \cos_{a(x)} x = 1 - \int_0^x\int_0^x a(x)dx^2 + \int_0^x\int_0^x a(x)dx^2 \int_0^x\int_0^x a(x)dx^2 - \cdots \tag{4.6}$$

Definition. The generalized sine is the solution of the differential equation (3.1) ($a(x) > 0$) with the initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 1. \tag{4.7}$$

I.e.

$$y_2(x) \equiv \sin_{a(x)} x = x - \int_0^x \int_0^x x a(x) dx^2 + \int_0^x \int_0^x a(x) dx^2 \int_0^x \int_0^x x a(x) dx^2 - \cdots \quad (4.8)$$

Now we will study the derivative of the solutions $\cos_{a(x)} x$ and $\sin_{a(x)} x$.

Theorem. The generalized cosine (4.6) satisfies the identity

$$\left(\cos_{a(x)} x\right)' = -\int_0^x a(x) \cos_{a(x)} x \, dx. \quad (4.9)$$

Proof. According to (4.6) we have

$$\begin{aligned} \left(\cos_{a(x)} x\right)' &= -\int_0^x a(x) dx + \int_0^x a(x) dx \int_0^x \int_0^x a(x) dx^2 - \cdots \\ &= -\int_0^x a(x) \left[1 - \int_0^x \int_0^x a(x) dx^2 + \int_0^x \int_0^x a(x) dx^2 \int_0^x \int_0^x a(x) dx^2 - \cdots\right]. \\ &= -\int_0^x a(x) \cos_{a(x)} x dx, \qquad \text{QED.} \end{aligned} \quad (4.10)$$

Similarly one can prove the following

Theorem. The generalized sine (4.8) satisfies the identity

$$\left(\sin_{a(x)} x\right)' = 1 - \int_0^x a(x) \sin_{a(x)} x \, dx. \quad (4.11)$$

Then we prove the validity of

Theorem. The generalized cosine and generalized sine are linearly independent solutions of differential equation (3.1).

Proof. The Wronskian of two solutions of the differential equation (3.1) is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2. \quad (4.12)$$

From the Liouville theorem of canonical differential equations of the second order it is valid that $W(x) = \text{const.}$, which we can choose to be $W(x = 0)$.

On the other hand from the relations (4.5) and (4.7) we have $y_1(0) = 1$, $y_1'(0) = 0$, $y_2(0) = 0$ and $y_2'(0) = 1$. Therefore, from the expression (4.12) we obtain that $W(0) = 1$, so the solutions y_1 and y_2 are linearly independent, QED.

The above theorem leads to the basic relation of generalized trigonometry of second order

$$\begin{vmatrix} \cos_{a(x)} x & \sin_{a(x)} x \\ \left(\cos_{a(x)} x\right)' & \left(\sin_{a(x)} x\right)' \end{vmatrix} = 1 \tag{4.13}$$

which in the special case $a(x) = 1$ transforms into the standard relation of the usual trigonometry

$$\cos^2 x + \sin^2 x = 1. \tag{4.14}$$

Analogously with usual trigonometry one can define generalized tangent and cotangent

$$\tan_{a(x)} x = \frac{\sin_{a(x)} x}{\cos_{a(x)} x}, \tag{4.15}$$

$$\cot_{a(x)} x = \frac{\cos_{a(x)} x}{\sin_{a(x)} x}, \tag{4.16}$$

which, of course, in the special case $a(x) = 1$ transforms into the usual tangent and cotangent.

In equation (3.1) let the coefficient be negative, i.e. $a(x) = -|a(x)|$. Then the series (3.7) and (3.8) converge uniformly and we can analogously with the general cosine and sine (4.6) and (4.8) define the general cosine hyperbolic and general sine hyperbolic

$$y_1(x) \equiv \cosh_{a(x)} x = 1 + \int_0^x\int_0^x |a(x)| dx^2 + \int_0^x\int_0^x |a(x)| dx^2 \int_0^x\int_0^x |a(x)| dx^2 - \cdots \tag{4.17}$$

$$y_2(x) \equiv \sinh_{a(x)} x = x + \int_0^x\int_0^x x|a(x)| dx^2 + \int_0^x\int_0^x |a(x)| dx^2 \int_0^x\int_0^x x|a(x)| dx^2 - \cdots \tag{4.18}$$

and also generalize tangent and cotangent hyperbolic

$$\tanh_{a(x)} x = \frac{\sinh_{a(x)} x}{\cosh_{a(x)} x}, \tag{4.19}$$

$$\coth_{a(x)} x = \frac{\cosh_{a(x)} x}{\sinh_{a(x)} x}. \tag{4.20}$$

Analogously with the Euler formula

$$e^x = \sinh x + \cosh x \tag{4.21}$$

one can define the generalized exponential function

$$e^x_{a(x)} = \sinh_{a(x)} x + \cosh_{a(x)} x \tag{4.22}$$

which the product $e^x_{a(x)} \cdot e^{-x}_{a(x)} \neq 1$, except when $a(x) = 1$.

Finally, we will prove the theorem of the parity of sine and cosine function.

Theorem. For cosine and sine functions the following identity is valid

$$\cos_{a(-x)}(-x) = \cos_{a(x)}(x), \qquad \sin_{a(-x)}(-x) = -\sin_{a(x)}(x). \tag{4.23}$$

The **Proof** is evident according to definition (4.6)

$$\begin{aligned} \cos_{a(-x)}(-x) = 1 - \int_0^{-x} dt \int_0^{-x} a(-t)dt \\ + \int_0^{-x} dt \int_0^{-x} a(-t)dt \int_0^{-x} dt \int_0^{-x} a(-t)dt - \cdots \end{aligned} \tag{4.24}$$

Then by the replacement $-t = q$ in the right hand side of (4.24) we have

$$\begin{aligned} \cos_{a(-x)}(-x) &= 1 - \int_0^x (-1)\, dq \int_0^x a(q)(-1)\, dq \\ &+ \int_0^x (-1)\, dq \int_0^x a(q)(-1)\, dq \int_0^x (-1)\, dq \int_0^x a(q)(-1)\, dq - \cdots \\ &= 1 - \int_0^x dq \int_0^x a(q)dq + \int_0^x dq \int_0^x a(q)dq \int_0^x dq \int_0^x a(q)dq - \cdots \\ &= \cos_{a(x)}(x). \end{aligned} \tag{4.25}$$

For sine function the proof is similar, see formula (4.8).

In the special case when the coefficient of the differential equation is an even function, i.e. $a(-x) = a(x)$ from the theorem it follows that the cosine solution is an even function and the sine solution is an odd function

$$\cos_{a(x)}(-x) = \cos_{a(x)}(x), \qquad \sin_{a(x)}(-x) = -\sin_{a(x)}(x). \tag{4.26}$$

That means the general solution (in a wider sense) of the canonical differential equation of the second order is a sum of odd functions and even functions.

It is evident that both parity relations (4.23) and (4.26) which are valid for sine function also are valid for tangent and cotangent functions.

In in the previous consideration we have introduced the generalized trigonometry of canonical differential equation of the second order. But these generalized trigonometric functions are formal. They are defined only via the multiple integrals (3.7), (3.8) etc. However, when we say "trigonometry" we always think of the triangle, cathetus, hypotenuse ... The question is: does there exist a connection between rectangle triangle and generalized trigonometric function?

The answer is affirmative [6].

First we define what is known as "phase" space as a two dimensional space where the abscissa and ordinate are a particular solution of the differential equation of the second order (3.1) $y_1(x)$ and $y_2(x)$. The linear independence of the solutions lead to the conclusion that the Wronscian (4.12) is equal to 1.

Now instead the orthogonal coordinates y_1 and y_2 we introduce the polar coordinates $\varrho(x)$ and $\varphi(x)$ such that

$$\begin{aligned} y_1(x) &= \varrho(x)\cos\varphi(x), \\ y_2(x) &= \varrho(x)\sin\varphi(x). \end{aligned} \tag{4.27}$$

Then from the condition $W(y_1, y_2) = 1$ and (4.12) we obtain

$$y_1 y_2' - y_1' y_2 = 1, \tag{4.28}$$

or if the equation (4.28) is transformed into the polar coordinates (4.27) it is valid that

$$\varrho^2 \frac{d\varphi}{dx} = 1. \tag{4.29}$$

After the integration of the expression (4.29) we have

$$x = \int_0^{\varphi} \varrho^2(\varphi)\, d\varphi = 2\mathcal{P}, \tag{4.30}$$

where $\mathcal{P}$ is called the sector surface (Fig. 4.1). So, if one represents the particular solutions y_1 and y_2 of the canonical differential equation as a curve in the phase space, the meaning of the variable x is a double sector surface. From the above formula it also follows that $d\mathcal{P}/dx = 1/2$. The sectorial velocity (in the phase space) is a constant! Does it resemble Kepler's second law? However Kepler's law of mechanics is a consequence of a motion in three dimensional space in the presence of a conservative central force. Our differential equation corresponds to dimensional harminic oscillator with time-dependent mass. It is not clear that there is an equivalence between these two processes.

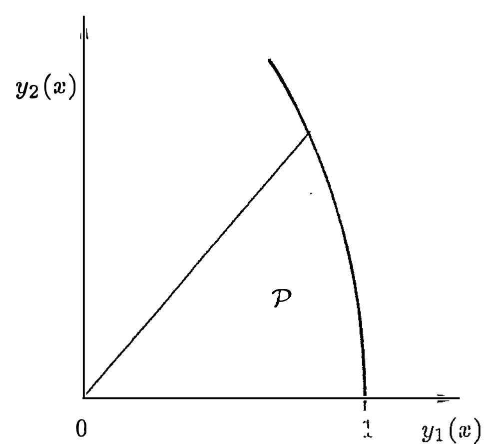

Fig. 4.1 The sector surface.

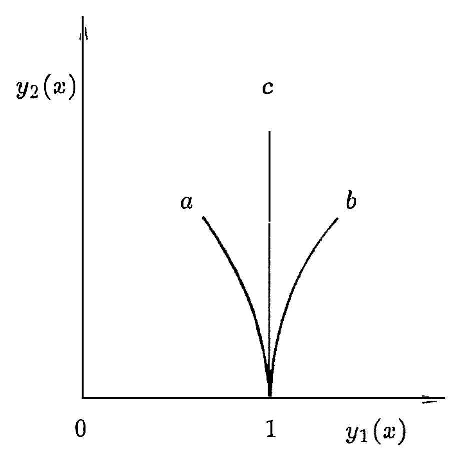

Fig. 4.2 The types of phase graphs.

In the special case when $a(x) = 1$, the particular solutions (3.7) and (3.8) become the usual cosine and sine functions: $y_1 = \cos x$, $y_2 = \sin x$. Then the phase trajectory is a circle with a radius $\varrho = 1$, and from the

integral (4.30) it follows that $x = \int_0^\varphi d\varphi = \varphi$. The meaning of the variable x is usual angle.

Fig. 4.2 shows the global behaviour of the phase trajectory of the canonical differential equations of the second order in the neighborhood of the point $(1, 0)$, for three types: a) elliptic type $a(x) > 0$, b) hyperbolic type $a(x) < 0$ and c) parabolic type $a(x) = 0$. The tangent on the phase trajectory in the point $(1, 0)$ is the vertical line. It can be proved simply by the calculation of the derivative and using formulae (4.9) and (4.11)

$$\begin{aligned}\lim_{x\to 0}\frac{dy_2}{dy_1} &= \lim_{x\to 0}\frac{(\sin_{a(x)} x)'}{(\cos_{a(x)} x)'} \\ &= \lim_{x\to 0}\frac{1-\int_0^x a(x)\sin_{a(x)} x}{-\int_0^x a(x)\cos_{a(x)} x} \sim \frac{1}{-0} \sim -\infty, \qquad \text{QED.}\end{aligned} \tag{4.31}$$

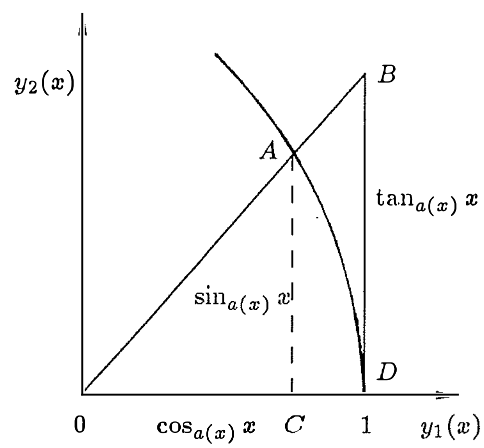

Fig. 4.3 Generalized trigonometry.

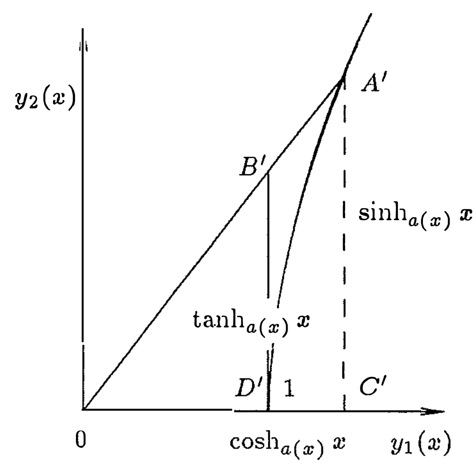

Fig. 4.4 Generalized hyperbolic trigonometry.

Now the geometrical interpretation of the generalized trigonometrical function has been shown in the Figs. 4.3 and 4.4. Namely, in Fig. 4.3 in the rectangular triangle OBD the sector line OC represents $\cos_{a(x)} x$, AC represents $\sin_{a(x)} x$ while BD represents $\tan_{a(x)} x$. On the other hand, in

the hyperbolic case (Fig. 4.4) the line segment OC' represents the function $\cosh_{a(x)} x$, $A'C'$ corresponds to $\sinh_{a(x)} x$, while $B'D'$ corresponds to $\tanh_{a(x)} x$.

The knowledge of phase trajectory can approach for one kind of inverse problem. Namely, if the form of phase graph $\varrho = \varrho(\varphi)$ is given, one can find [6] the coefficient $a(x)$ and particular solutions of the canonical differential equation.

5. Canonical Equation of n-th Order

The equation

$$y^{(n)} + a(x)y = 0 \tag{5.1}$$

is known as a canonical differential equation of n-th order. Repeating all previoũs consideration gives the

Theorem. The differential equation (5.1) has a particular integral

$$y_1(x) = 1 - \underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} a(x)\,dx^n + \underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} a(x)\,dx^n \underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} a(x)\,dx^n - \cdots \tag{5.2}$$

Proof. After differentiation of the expression (5.2) n times we have

$$\begin{aligned} y_1^{(n)} &= -a(x) + a(x)\underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} a(x)\,dx^n \\ &- a(x)\underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} a(x)\,dx^n \underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} a(x)\,dx^n + \cdots \\ &= -a(x)\left[1 - \underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} a(x)\,dx^n + \underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} a(x)\,dx^n \underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} a(x)\,dx^n - \cdots\right] \end{aligned} \tag{5.3}$$

$$= -a(x)y_1,$$

i.e.

$$y_1^{(n)} + a(x)y_1 = 0, \qquad \text{QED.} \tag{5.4}$$

By analogy, it can be proved that the functions

$$y_2(x) = x - \underbrace{\int_0^x\int_0^x \cdots \int_0^x}_{n} xa(x)\,dx^n + \underbrace{\int_0^x\int_0^x \cdots \int_0^x}_{n} a(x)\,dx^n \underbrace{\int_0^x\int_0^x \cdots \int_0^x}_{n} xa(x)\,dx^n - \cdots, \tag{5.5}$$

$$y_3(x) = x^2 - \underbrace{\int_0^x\int_0^x \cdots \int_0^x}_{n} x^2a(x)\,dx^n + \underbrace{\int_0^x\int_0^x \cdots \int_0^x}_{n} a(x)\,dx^n \underbrace{\int_0^x\int_0^x \cdots \int_0^x}_{n} x^2a(x)\,dx^n - \cdots, \tag{5.6}$$

$$\cdots$$

$$y_n(x) = x^{n-1} - \underbrace{\int_0^x\int_0^x \cdots \int_0^x}_{n} x^{n-1}a(x)\,dx^n + \underbrace{\int_0^x\int_0^x \cdots \int_0^x}_{n} a(x)\,dx^n \underbrace{\int_0^x\int_0^x \cdots \int_0^x}_{n} x^{n-1}a(x)\,dx^n - \cdots, \tag{5.7}$$

are the particular integrals of differential equation (5.1). The above particular integrals can be written in the closed form [5]

$$y_1(x) = \sum_{k=0}^{\infty}(-1)^k \underbrace{\underbrace{\int_0^x\int_0^x \cdots \int_0^x}_{n} a(x)\,dx^n \cdots \underbrace{\int_0^x\int_0^x \cdots \int_0^x}_{n} a(x)\,dx^n}_{\text{k } n\text{-times integrals}},$$

$$y_2(x) = \sum_{k=0}^{\infty}(-1)^k \underbrace{\underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} a(x)\,dx^n \cdots \underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} xa(x)\,dx^n}_{\text{k } n\text{-times integrals}},$$

$$\vdots \tag{5.8}$$

$$y_n(x) = \sum_{k=0}^{\infty}(-1)^k \underbrace{\underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} a(x)\,dx^n \cdots \underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{n} x^{n-1}a(x)\,dx^n}_{\text{k } n\text{-times integrals}}.$$

The characteristics of solutions (5.8) is that in the neighborhood of the point $x = 0$ they have the behavior of a integer degree: $y_1(x) \sim 1$, $y_2(x) \sim x$, $y_3(x) \sim x^2$, $\cdots$, $y_n(x) \sim x^{n-1}$.

Theorem. The system of the functions (5.8) is a fundamental system of particular integrals of canonical equation (5.1).

Proof. The Wronskian of solutions (5.8) is

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}. \tag{5.9}$$

If we denote $W_0 = W[y_1(0), y_2(0), \ldots, y_n(0)]$ then by an elementary approach to the formula (5.8) there follows the result

$$W_0 = \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1! & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2! & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & (n-1)! \end{vmatrix} = (n-1)!\cdots 2!\,1! \neq 0 \qquad \text{QED.} \tag{5.10}$$

So, the general solution $y(x)$ of the canonical differential equation of n-th order is given by the formula

$$y(x) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n, \tag{5.11}$$

where C_i, $i = 1, 2, \ldots, n$, are arbitrary constants, while y_i are the fundamental solutions (5.8).

Theorem. The series (5.8) are convergent uniformly for every analytical coefficient $a(x)$.

Proof. From $|a(x)| < M$ we have that

$$|y_1| < 1 + M\frac{|x|^n}{n!} + M^2\frac{|x|^{2n}}{2n!} + \cdots = \sum_{k=1}^{\infty} \frac{\left(\sqrt[n]{M}|x|\right)^{nk}}{(nk)!},$$

$$|y_2| < |x| + M\frac{|x|^{n+1}}{(n+1)!} + M^2\frac{|x|^{2n+1}}{(2n+1)!} + \cdots = \sum_{k=1}^{\infty} \frac{1}{\sqrt[n]{M}} \frac{\left(\sqrt[n]{M}|x|\right)^{nk+1}}{(nk+1)!},$$

$$\vdots$$

$$\begin{aligned} |y_n| &< |x|^{n-1} + M\frac{|x|^{2n-1}}{(2n-1)!} + M^2\frac{|x|^{3n-1}}{(3n-1)!} + \cdots \\ &= |x|^{n-1} + \sum_{k=1}^{\infty} \frac{1}{\left(\sqrt[n]{M}\right)^{n-1}} \frac{\left(\sqrt[n]{M}|x|\right)^{n(k+1)-1}}{(n(k+1)-1)!}, \end{aligned} \tag{5.12}$$

and because the majoring series is convergent (the D'Alamberts test), the series (5.8) converge uniformly, QED.

Similarly to previous considerations, we can study the phase space of the canonical differential equation of n-th order (5.1). It is an n- dimensional Euclidean space of the particular solutions (5.8). Assuming the constants C_1, C_2, ... C_n in (5.11) as scale factors one can represent the solution of differential equation (5.1) by the point in this n dimensional space. Then the evolution of the solution $y(x)$ is a curve in this space.

The elementary n-volume of the n-body constructed on the curve $\vec{r}(x)$ in n dimensional phase space is

$$d\,V^{(n)} = \frac{1}{2} W[y_1(x), y_2(x), \ldots, y_n(x)]\, dx. \tag{5.13}$$

where W is the Wronskian (5.9). Then from the relations (5.10) and (5.13) we obtain the geometrical interpretation of the parameter x of the canonical differential equation of n-th order [6]

$$x = \frac{2V^{(n)}}{\prod_{i=1}^{n-1} i!}. \tag{5.14}$$

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